

JUL 5 1929

AMERICAN JOURNAL OF MATHEMATICS

FOUNDED BY THE JOHNS HOPKINS UNIVERSITY

EDITED BY

E. W. CHITTENDEN
UNIVERSITY OF IOWA

ABRAHAM COHEN
THE JOHNS HOPKINS UNIVERSITY

A. B. COBLE
UNIVERSITY OF ILLINOIS

G. C. EVANS
RICE INSTITUTE

FRANK MORLEY, CHAIRMAN
THE JOHNS HOPKINS UNIVERSITY

WITH THE COÖPERATION OF

E. T. BELL

F. D. MURNAGHAN

W. A. MANNING

HARRY BATEMAN

J. R. KLINE

E. P. LANE

HARRY LEVY

MARSTON MORSE

PUBLISHED UNDER THE JOINT AUSPICES OF

THE JOHNS HOPKINS UNIVERSITY

AND

THE AMERICAN MATHEMATICAL SOCIETY

Volume LI, Number 3

JULY, 1929

THE JOHNS HOPKINS PRESS
BALTIMORE, MARYLAND
U. S. A.

CONTENTS

A Class of Polynomials and Rational Functions in Four Variables. By E. T. BELL,	329
Representations in the Form $xy + yz + zx$. By WALTER H. GAGE, . .	345
The Resultant of Two Power Series in Two Variables. By L. L. DINES, .	349
An Analysis of Logical Substitution. By H. B. CURRY,	363
On Extending a Correspondence in the Sense of Antoine. By HARRY MERRILL GEHMAN,	385
Green's Functions for Differential Systems Containing a Parameter. By W. W. ELLIOTT,	397
A Class of Invariant Functionals of Quadratic Functional Forms. By T. S. PETERSON,	417
On the Contact of a Quartic Surface with an Analytic Surface. By ERNEST P. LANE,	431
Non-Monoidal Involutions Having a Congruence of Invariant Conics. By HAZEL EDITH SCHOONMAKER,	439
Extensions of Clifford's Chain-Theorem. By F. MORLEY,	465
An Extension to Clifford's Chain. By PAUL SMITH WAGNER,	473
On the Group for a Class of Self-Dual Plane Rational Curves. By LUTHER E. WEAR,	482
Determination of All the Abstract Groups of Order 72. By G. A. MILLER,	491

THE AMERICAN JOURNAL OF MATHEMATICS will appear four times yearly.

The subscription price of the JOURNAL for the current volume is \$7.50 (foreign postage 25 cents); single numbers \$2.00.

A few complete sets of the JOURNAL remain on sale.

Papers intended for publication in the JOURNAL may be sent to any of the Editors.

Editorial communications may be sent to Dr. A. COHEN at the Johns Hopkins University.

Subscriptions to the JOURNAL and all business communications should be sent to THE JOHNS HOPKINS PRESS, BALTIMORE, MARYLAND, U. S. A.

Entered as second-class matter at the Baltimore, Maryland, Postoffice, acceptance for mailing at special rate of postage provided for in Section 1103, Act of October 3, 1917, Authorized on July 3, 1918

A Class of Polynomials and Rational Functions in Four Variables.

BY E. T. BELL.

1. *Introduction.* Let (w, x, y, z) be the coordinates of the quaternion q ,

$$(1.1) \quad q \equiv w + ix + jy + kz,$$

where w, x, y, z are independent variables, real or complex, and let m be an integer. Then q^m is a quaternion; let its coordinates be (W_m, X_m, Y_m, Z_m) , where

$$(1.2) \quad T_m \equiv T_m(w, x, y, z) \quad (T = W, X, Y, Z).$$

If $m \geq 0$, T_m is a homogeneous polynomial of degree m in w, x, y, z . If n is an integer ≥ 0 , we shall see that

$$(1.3) \quad W_{-n} = W_n/N^n, \quad T_{-n} = -T_n/N^n \quad (T = X, Y, Z),$$

where N is the norm of q ,

$$(1.4) \quad \begin{aligned} N &\equiv N(w, x, y, z) \equiv w^2 + x^2 + y^2 + z^2, \\ N^n &\equiv W_n^2 + X_n^2 + Y_n^2 + Z_n^2; \end{aligned}$$

and hence, if $m < 0$, $T_m(T = W, X, Y, Z)$ is a rational function of degree m and homogeneous. Thus T_m in all cases is homogeneous of degree m , where m is an integer ≥ 0 .

We shall see also that

$$(1.5) \quad T_{m+2} - 2wT_{m+1} + NT_m = 0 \quad (T = W, X, Y, Z),$$

for all integers m . Let S be the tensor of q ; then

$$(1.6) \quad S \equiv S(w, x, y, z) \equiv x^2 + y^2 + z^2.$$

Let α, β ,

$$(1.7) \quad \alpha \equiv \alpha(w, x, y, z), \quad \beta \equiv \beta(w, x, y, z)$$

be the roots of

$$(1.8) \quad \theta^2 - 2w\theta + N = 0.$$

Then we may take

$$(1.9) \quad \alpha = w + iS^{1/2}, \quad \beta = w - iS^{1/2},$$

where i is the imaginary unit of algebra [there can be no confusion between this and i as in (1.1)], and any solution of (1.5) is of the form

$$(1.10) \quad T_n = a\alpha^n + b\beta^n \quad (n = 0, \pm 1, \pm 2, \dots),$$

where a, b are arbitrary constants. The particular pair U_n, V_n of solutions

$$(1.11) \quad U_n \equiv (\alpha^n - \beta^n)/(\alpha - \beta), \quad V_n \equiv \alpha^n + \beta^n,$$

whose initial values are $(U_0, U_1) = (0, 1)$, $(V_0, V_1) = (2, 2w)$, will be required, as in terms of them the T_n are simply expressible. The U_n, V_n are the well known "singly periodic numerical functions" of E. Lucas,* for the parameter $(2w, N)$; that is, his P, Q are our $2w, N$ respectively. From (1.8), (1.11) we have

$$(1.12) \quad U_{-n} = -N_n/N^n, \quad V_{-n} = V_n/N^n,$$

for all integers n . When necessary we shall indicate the variables w, x, y, z , of which U_n, V_n are functions, by writing

$$(1.13) \quad T_n \equiv T_n(w, x, y, z) \quad (T = U, V).$$

The polynomials and rational functions defined in (1.2) are those to be discussed. They have many interesting properties. Thus, their close connection with the Lucas functions for the parameter $(2w, N)$ enables us to transfer directly a great part of Lucas' theory to the T_n . Again, the T_m , $m > 0$, are Appell polynomials in w , but not in x, y , or z ; with respect to a properly chosen argument they can be exhibited as Tchebycheff polynomials; they are resolvable, in an elegant way, into factors of the first degree; with respect to w, x, y, z the $T_n (n \geq 0)$ are solutions of an interesting set of self-adjoint linear differential equations of the second order. When $N = 1$, there is an isomorphism between the algebraic properties of the $T_n (T = W, X, Y, Z)$ and the σ, ζ, p of Weierstrass.

There is an immediate generalization of the T_n , obtained by replacing n in (1.11) by a real or complex variable. Practically all of the algebra developed here for the T_n goes over with but slight changes, if any, to the generalization; but as the complete development requires a revision of certain parts of Lucas' theory, we shall defer discussion of it to another occasion, and restrict m in (1.2) to be an integer. A similar generalization is obviously feasible from the differential equations for the polynomials, precisely as in passing from the Bessel coefficients to the Bessel functions.

* *American Journal of Mathematics*, Vol. 1 (1878), pp. 184, 289.

2. The T_n ($T = W, X, Y, Z$). Throughout this section n is an integer ≥ 0 and q, q' are quaternions. Let the coordinates of q, q' be $(w, x, y, z), (w', x', y', z')$ respectively, and let the coordinates of qq' be (W, X, Y, Z) . Then

$$(2.1) \quad \begin{aligned} W &= ww' - xx' - yy' - zz', \\ X &= wx' + xw' + yz' - zy', \\ Y &= wy' + yw' + zx' - xz', \\ Z &= wz' + zw' + xy' - yx'. \end{aligned}$$

Let the coordinates of q^n be (W_n, X_n, Y_n, Z_n) . Then, multiplication of quaternions being associative, $q^n q = qq^n = q^{n+1}$, and hence, by (2.1), we have

$$(2.2) \quad zY_n - yZ_n = xZ_n - zX_n = yX_n - xY_n = 0,$$

and

$$(2.3) \quad \begin{aligned} W_{n+1} &= wW_n - xX_n - yY_n - zZ_n, \\ X_{n+1} &= xW_n + wX_n, \\ Y_{n+1} &= yW_n + wY_n, \\ Z_{n+1} &= zW_n + wZ_n. \end{aligned}$$

From (2.2), (2.3), by a short calculation which need not be reproduced, we have

$$(2.4) \quad T_{n+2} - 2wT_{n+1} + NT_n = 0 \quad (T = W, X, Y, Z);$$

and from q^0, q , we see that the pairs of initial values which, with (2.4) completely define the T_n for $n = 0, 1, 2, \dots$, are

$$(2.5) \quad \begin{aligned} (W_0, W_1) &= (1, w), & (X_0, X_1) &= (0, x), \\ (Y_0, Y_1) &= (0, y), & (Z_0, Z_1) &= (0, z). \end{aligned}$$

It is clear that T_n is a homogeneous polynomial of degree n in w, x, y, z .

Denote the reciprocal of q by q^{-1} , so that $qq^{-1} = q^{-1}q = 1$,

$$(2.6) \quad (q^{-1})^n q^n = q^n (q^{-1})^n = (qq^{-1})^n = 1.$$

Let the coordinates of q^{-1} be (w', x', y', z') . Then

$$(2.7) \quad (w', x', y', z') = (w/N, -x/N, -y/N, -z/N).$$

Define the $T_{-n} \equiv T_{-n}(w, x, y, z)$ by

$$(2.8) \quad (q^{-1})^n = W_{-n} + iX_{-n} + jY_{-n} + kZ_{-n};$$

namely, the coordinates of $(q^{-1})^n$ are $(W_{-n}, X_{-n}, Y_{-n}, Z_{-n})$. Write

$$(2.9) \quad T_n' \equiv T(w', x', y', z') \quad (T = W, X, Y, Z).$$

Then $T_n' = T_{-n}$, or in full,

$$(2.10) \quad T_n(w/N, -x/N, -y/N, -z/N) = T_{-n}(w, x, y, z).$$

From (2.6) we readily infer by (2.1) that

$$(2.11) \quad W_{-n} = W_n/N^n, \quad T_{-n} = -T_n/N^n \quad (T = X, Y, Z).$$

Solving (2.11) for W_n, T_n , we see that the result could be obtained by changing the sign of n in (2.11). Hence the theorem (1.3) is proved.

The theorem (1.5) follows at once from (1.3) and (2.4).

By (1.3) we may remove from the integer n in (2.2) the restriction that it be ≥ 0 ; n in (2.2) is therefore now any integer. Similarly for (2.3) and (2.4).

Hence finally the integer $n \geq 0$ in all formulas concerning T_n ($T = W, X, Y, Z$) in this section can be replaced by the integer $m \leq 0$.

3. *The functions U_n, V_n ($n \leq 0$).* As already defined in (1.11), (1.13), U_n, V_n are the Lucas functions for the parameter $(2w, N)$. If now Θ_n ($n = 0, 1, 2, \dots$) is any solution of

$$(3.1) \quad \Theta_{n+2} - 2w\Theta_{n+1} + N\Theta_n = 0,$$

we can express Θ_n as a linear homogeneous function of the linearly independent solutions of U_n, V_n . We find

$$(3.2) \quad \Theta_n = (\Theta_1 - w\Theta_0)U_n + \frac{1}{2}\Theta_0 V_n.$$

Hence, by (2.5), we have

$$(3.3) \quad W_n = \frac{1}{2}V_n, \quad X_n = xU_n, \quad Y_n = yU_n, \quad Z_n = zU_n,$$

for all integers n .

Thus in any formula or theorem concerning the Lucas functions U_n, V_n for the parameter $(2w, N)$, we may replace U_n by any one of $X_n/x, Y_n/y, Z_n/z$, and V_n by $2W_n$, to obtain a formula or theorem for the T_n ($T = W, X, Y, Z$).

From (1.2), (1.6), (1.7) we have the following parity properties of the functions α, β, U_n, V_n ,

$$(3.4) \quad \begin{aligned} \alpha(-w, x, y, z) &= -\beta(w, x, y, z), \\ \alpha(w, -x, -y, -z) &= \alpha(w, x, y, z), \\ \beta(w, -x, -y, -z) &= \beta(w, x, y, z); \end{aligned}$$

$$(3.5) \quad \begin{aligned} U_n(-w, x, y, z) &= (-1)^{n-1}U_n(w, x, y, z), \\ V_n(-w, x, y, z) &= (-1)^n V_n(w, x, y, z); \\ U_n(w, -x, -y, -z) &= U_n(w, x, y, z), \\ V_n(w, -x, -y, -z) &= V_n(w, x, y, z); \end{aligned}$$

and therefore by (3.3),

$$(3.6) \quad \begin{aligned} W_n(-w, x, y, z) &= (-1)^n W_n(w, x, y, z), \\ W_n(w, -x, -y, -z) &= W_n(w, x, y, z); \end{aligned}$$

$$(3.7) \quad \begin{aligned} T_n(-w, x, y, z) &= (-1)^{n-1} T_n(w, x, y, z), \\ T_n(w, -x, -y, -z) &= -T(w, x, y, z), \\ (T &= X, Y, Z). \end{aligned}$$

4. *Explicit Forms.* Let n be an integer ≥ 0 , and define

$$(4.1) \quad \phi_n \equiv \phi_n(w, x, y, z), \quad \psi_n \equiv \psi_n(w, x, y, z)$$

by

$$(4.2) \quad \phi_{2n} = \psi_{2n+1} = 0, \quad \phi_{2n+1} = \psi_{2n} = (-S)^n,$$

where S is as in (1.6). Then, from (1.11), (1.9) we have

$$(4.3) \quad U_n = (w + \phi)^n, \quad V_n = 2(w + \psi)^n;$$

and therefore, by (3.3),

$$(4.4) \quad \begin{aligned} W_n &= (w + \psi)^n, & X_n &= x(w + \phi)^n, \\ Y_n &= y(w + \phi)^n, & Z_n &= z(w + \phi)^n, \end{aligned}$$

in which, after expansion by the binomial theorem ψ^r, ϕ^r ($r = 0, \dots, n$) are to be replaced by ψ_r, ϕ_r respectively, as always in the symbolic or umbral calculus. Together, (4.4) and (1.3) give the explicit forms of the T_m for all integers m .

Following Lucas (*loc. cit.*, p. 189), we can express U_n, V_n in terms of circular or hyperbolic functions of an appropriate argument. Principal determinations of all irrationals are to be understood. We find

$$(4.5) \quad \begin{aligned} U_n &= (N^n/S)^{1/2} \sin \left\{ ni \log \left(\frac{w - iS^{1/2}}{N^{1/2}} \right) \right\}, \\ V_n &= 2N^{1/2} \cos \left\{ ni \log \left(\frac{w + iS^{1/2}}{N^{1/2}} \right) \right\}. \end{aligned}$$

From (3.3) we may write down by (4.5) the trigonometric forms of the T_n . It is unnecessary to transcribe them. The above have been reduced. By means of the following, equivalent to (4.5) and its consequences, the whole of trigonometry can be transposed to relations between the T_n . Write

$$(4.6) \quad \lambda \equiv \lambda(w, x, y, z) \equiv (i/2) \log (\alpha/\beta).$$

Then, from (4.5) we have

$$(4.7) \quad \sin n\lambda = - (S/N^n)^{1/2} U_n, \quad \cos n\lambda = 1/2 N^{-n/2} V_n;$$

$$(4.8) \quad \sin n\lambda = - (S/N^n)^{1/2} \cdot T_n/t, \quad \cos n\lambda = (1/N^n)^{1/2} W_n, \\ (T, t) = (X, x), (Y, y), (Z, z).$$

As in previous sections it follows at once that the restriction $n \geq 0$ can be removed from (4.5)-(4.8), and that these results are valid for all integers n .

The connection with Tchebycheff polynomials is now evident. As we have already used T, U , the customary letters for these polynomials, we shall define the Tchebycheff polynomials $L_n(u)$, $M_n(u)$, first for n an integer ≥ 0 , (L, M here replace T, U in the usual notation) by

$$(4.9) \quad \cos \lambda \equiv u, \quad L_n(u) \equiv \cos n\lambda, \quad \sin \lambda = (1 - u^2)^{1/2}, \\ M_n(u) \equiv \frac{1}{n+1} \frac{d}{du} L_{n+1}(u) = \frac{\sin (n+1)\lambda}{\sin \lambda},$$

so that $L_m(u)$ is a polynomial of degree m in u in which the coefficient of u^m is 2^{m-1} , and $M_m(u)$ is a polynomial of degree m in u , with leading coefficient 2^m ($m = 1, 2, \dots$). From (4.7)-(4.9) we have

$$(4.10) \quad T_{n+1} = -t [N^{n+1} S^{-1} (1 - u^2)]^{1/2} M_n(u), \\ (T, t) = (U, 1), (X, x), (Y, y), (Z, z);$$

$$(4.11) \quad T_n = t N^{n/2} L_n(u), \quad (T, t) = (V, 2), (W, 1),$$

which, so far, have been proved only for $n = 0, 1, \dots$. But, taking in (4.9) only the second form of the definition of $M_n(u)$, we see that the definitions,

$$(4.12) \quad L_{-n}(u) \equiv L_n(u), \quad M_{-n}(u) \equiv -M_{n-2}(u),$$

the second of which implies $M_{-1}(u) = 0$, are consistent with everything that precedes; so that finally, as in previous extensions from $n \geq 0$ to $n \leq 0$, we see that (4.10), (4.11) hold for all integers $n \leq 0$.

5. *Algebraic relations between the T_n .* An indefinite number of algebraic relations between the T_n ($T = W, X, Y, Z$) can be developed isomorphically with the theory of the circular or the hyperbolic functions, either from Lucas' functions or directly from (4.8) or its equivalent in terms of hyperbolic functions. Write

$$(5.1) \quad \mu \equiv \mu(w, x, y, z) = -i\lambda,$$

with λ as in (4.6). Then

$$(5.2) \quad \operatorname{sh} n\mu = i(S/N^n)^{1/2} U_n, \quad \operatorname{ch} n\mu = 1/2 N^{-n/2} V_n;$$

$$(5.3) \quad \begin{aligned} \operatorname{sh} n\mu &= i(S/N^n)^{1/2} \cdot T_n/t, & \operatorname{ch} n\mu &= (1/N^n)^{1/2} W_n, \\ (T, t) &= (X, x), (Y, y), (Z, z), \end{aligned}$$

for all integers n .

The number of relations obtainable being unlimited, there is no point in writing out an extensive selection, as all follow, without computations, by the means indicated. We may, however, notice a few, some of which will be found useful in verifying an algebraic isomorphism with the elliptic and allied functions of Weierstrass. In what follows, m, n are arbitrary integers.

Corresponding to $ch^2\theta + sh^2\theta = 1$, we have

$$(5.4) \quad t^2 W_n^2 + S T_n^2 = t^2 N^n,$$

for the same (T, t) as in (5.3). The addition and subtraction theorems for the T_n ($T = W, X, Y, Z$) with respect to n are

$$(5.5) \quad \begin{aligned} T_{m+n} &= T_m W_n + W_m T_n, \\ N^n T_{m-n} &= T_m W_n - W_m T_n, \end{aligned}$$

for (T, t) as in (5.3); for the same (T, t) ,

$$(5.6) \quad \begin{aligned} t^2 W_{m+n} &= t^2 W_m W_n - S T_m T_n, \\ T^2 N^n W_{m-n} &= t^2 W_m W_n + S T_m T_n. \end{aligned}$$

The addition theorems can of course be obtained from $q^m q^n = q^{m+n}$ and (2.1), (2.2); the subtraction theorems then follow by (1.3). If this method be used, we find

$$W_{m+n} = W_m W_n - X_m X_n - Y_m Y_n - Z_m Z_n,$$

which, by (3.3), is equivalent to the first of (5.6).

From (5.5) we get an important identity, valid only when $N=1$, $T=X, Y, Z$:

$$(5.7) \quad (W_m/T_m)^2 - (W_n/T_n)^2 = - (T_{m+n} T_{m-n} / T_m^2 T_n^2).$$

The equivalent of De Moivre's theorem has many interesting consequences, of which we record only the following multiplication theorems. Let r be an integer ≥ 0 . Then, from (3.3) and

$$2\alpha^n = V_n + 2iS^{1/2}U_n,$$

we infer, for $(T, t) = (X, x), (Y, y), (Z, z)$, that

$$(tW_n + iS^{1/2}T_n)^m = t^{m-1}(tW_{nm} + iS^{1/2}T_{nm}),$$

for all integers n, m . Take $m=r$. Then we find at once

$$(5.8) \quad t^r W_{nr} = (tW_n + \psi T_n)^r, \quad t^{r-1} T_{nr} = (tW_n + \phi T_n)^r,$$

where ϕ, ψ are as in (4.2), valid for all integers n and for all integers $r \geq 0$. Note that the zeroth power of a binomial involving ϕ or ψ is ϕ_0 or ψ_0 .

We shall pass over the numerous algebraic properties that parallel those of Lucas' functions, such as the expression of the polynomials as recurrences, or as terminating continued fractions, their remarkable divisibility properties, and note only two arithmetical theorems. From identities such as (5.4) we infer parametric solutions of diophantine equations. Thus, if in (5.4), n be restricted to be ≥ 0 , and if k is an integer > 0 not of the form $4^h(8s+7)$, $h, s \geq 0$, we find immediately at least 12 $E(k)$ sets $(\xi, \eta, \zeta, \theta)$, of solutions, where $E(k)$ is the usual class number function, of

$$\xi^2 + k\eta^2 = \zeta^2\theta^n,$$

where k, n are given constant integers of the kinds prescribed. Again, if p is a prime > 1 of the form $4h+1$, the quotient $4T_{pr}/T_r$ ($T = X, Y, Z$), where r is an integer > 0 , is expressible in the form $P^2 - pQ^2$, where P, Q are polynomials in N and functions $2W_j$, for determinate values of $j \geq 0$, with integer coefficients. This is but one of several of a similar kind. Thus, for primes $p = 4h+3$, the form is of the type $-4SP^2 + pQ^2$. These follow from Lucas' expression (*loc. cit.*, p. 227) of $4U_{pr}/U_r$ as a cyclotomic quadratic form.

6. *Linear relations.* The following are useful in computations with the T_n . Let n be an integer ≥ 0 , unless otherwise stated. The general solution Θ_n of (3.1) may be written in the form

$$(6.1) \quad (\alpha - \beta)\Theta_n = (\Theta_1 - \beta\Theta_0)\alpha^n + (\alpha\Theta_0 - \Theta_1)\beta^n,$$

where Θ_0, Θ_1 are arbitrary constants. Hence

$$(6.2) \quad (\alpha - \beta)N^n\Theta_{-n} = (\alpha\Theta_0 - \Theta_1)\alpha^n + (\Theta_1 - \beta\Theta_0)\beta^n,$$

and therefore, on multiplication and reduction,

$$(6.3) \quad 4SN^n\Theta_n\Theta_{-n} = (\Theta_1^2 - 2w\Theta_1\Theta_0 + N\Theta_0^2)V_{2n} - 2[\Theta_1^2 - 2w\Theta_0\Theta_1 + (w^2 - S)\Theta_0^2]N^n;$$

also, by addition,

$$(6.4) \quad \Theta_n + N^n\Theta_{-n} = \Theta_0V_n.$$

Hence, if $\Theta_0 = 0$,

$$(6.5) \quad \Theta_{-n} = -\Theta_n/N^n \quad (\Theta_0 = 0),$$

which includes the results previously stated for U, X, Y, Z ; that for W is included in (6.4), by (3.3).

Let Θ_n, Φ_n be linearly independent solutions of (3.1), so that

$$(6.6) \quad D \equiv \Theta_0 \Phi_1 - \Phi_0 \Theta_1 \neq 0.$$

Then we can express any solution A_n as a linear homogeneous function of Θ_n, Φ_n . We find

$$(6.7) \quad DA_n = (A_0 \Phi_1 - A_1 \Phi_0) \Theta_n + (A_1 \Theta_0 - A_0 \Theta_1) \Phi_n.$$

The necessary initial values for (6.7) and subsequent equations are

$$(6.8) \quad \begin{array}{lll} U_0 = 0, & U_1 = 1, & U_2 = 2w; \\ V_0 = 2, & V_1 = 2w, & V_2 = 2(w^2 - S); \\ W_0 = 1, & W_1 = w, & W_2 = w^2 - S; \\ T_0 = 0, & T_1 = t, & T_2 = 2wt, \\ (T, t) = (X, x), (Y, y), (Z, z). \end{array}$$

Hence the condition (6.6) is satisfied only for the eight pairs (Θ, Φ) as follows,

$$\begin{array}{llll} (U, V), & (U, W), & (V, X), & (V, Y), \\ (V, Z), & (X, W), & (Y, W), & (Z, W). \end{array}$$

Since $V_n = 2W_n$, it suffices to consider only v_n or W_n in the consequences of (6.7), (6.8), say W_n . Then, expressing any solution A_n of (3.1) in terms of the W_n, X_n, Y_n, Z_n , we find

$$(6.9) \quad A_n = (A_1 - wA_0) U_n + A_0 W_n;$$

$$(6.10) \quad tA_n = (A_1 - wA_0) T_n + tA_0 W_n, \quad (T, t) = (X, x), (Y, y), (Z, z),$$

either of which implies the other by (3.3). Thus there is but one relation necessary, say (6.9). By (6.8) we readily check (6.9) on (3.3).

A useful alternative to (6.9) is the expression of A_{n+1} as a linear homogeneous function of Θ_{n+1}, Θ_n , where A_n, Θ_n are any solutions of (3.1):

$$(6.11) \quad (\Theta_1^2 - \Theta_0 \Theta_2) A_{n+1} = (A_1 \Theta_1 - A_2 \Theta_0) \Theta_{n+1} + (A_2 \Theta_1 - A_1 \Theta_2) \Theta_n.$$

Hence, by (6.8), we have

$$(6.12) \quad \begin{array}{l} tA_n = A_1 T_n + (A_2 - 2wA_1) T_{n-1}, \\ (T, t) = (U, 1), (X, x), (Y, y), (Z, z); \end{array}$$

$$(6.13) \quad \begin{array}{l} tSA_n = (wA_1 - A_2) T_n + [wA_2 - (w^2 - S)A_1] T_{n-1}, \\ (T, t) = (W, 1), (V, 2). \end{array}$$

These give the following, which, with (3.3), are all of the types (6.12), (6.13) for the W, X, Y, Z ,

$$\begin{aligned}
 (6.14) \quad ST_n &= -t(wW_n - NW_{n-1}) = t(wW_n - W_{n+1}), \\
 tW_n &= wT_n - NT_{n-1} = T_{n+1} - wT_n, \\
 (T, t) &= (X, x), (Y, y), (Z, z).
 \end{aligned}$$

Similarly, and using (3.1) with $\Theta = U, V$, we find

$$(6.15) \quad 2SU_n = -wV_n + NV_{n-1} = wV_n - V_{n+1},$$

$$(6.16) \quad V_n = 2(wU_n - NU_{n-1}) = 2(-wU_n + U_{n+1}).$$

7. *Derivatives.* It will be evident that the total derivatives in all that follows, may, if desired, be replaced by partial derivatives. There is no gain in generality if partials be adopted. From (1.6), (1.9) we have

$$(7.1) \quad (d\alpha/dw) = (d\beta/dw) = 1,$$

$$(7.2) \quad S^{1/2}(d\alpha/dt) = -S^{1/2}(d\beta/dt) = it \quad (t = x, y, z).$$

Since $\alpha - \beta = 2iS$ is independent of w , it follows from (1.11), (3.3) that

$$(7.3) \quad (dT_n/dw) = nT_{n-1} \quad (T = U, V, W, X, Y, Z),$$

for all integers n . Hence all the above six T_n are Appell polynomials in w . Otherwise, for $n > 0$, (7.3) is obvious from (4.3), (4.4); and for $n < 0$, (7.3) then follows from the same source on using (1.3).

From (7.1), (7.2), (1.11) we see that, for all integers n , and $t = x, y, z$,

$$(7.4) \quad 2S(dU_n/dt) = t(nV_{n-1} - 2U_n^*),$$

$$(7.5) \quad (dV_n/dt) = -2ntU_{n-1};$$

and hence, by (6.15), (6.16),

$$(7.6) \quad S(dU_n/dt) = t[(n-1)U_n - nwU_{n-1}],$$

$$(7.7) \quad S(dT_n/dt) = nt(T_n - wT_{n-1}) \quad (T = V, W).$$

The following are immediate from (3.3), (7.6):

$$\begin{aligned}
 (7.8) \quad S(dT_n/dt) &= t[(n-1)T_n - nwT_{n-1}], \\
 (T, t) &= (X, y), (X, z), (Y, z), (Y, x), (Z, x), (Z, y),
 \end{aligned}$$

valid for all integers n . There remain to be calculated only the dT_n/dt for $(T, t) = (X, x), (Y, y), (Z, z)$. For these (T, t) we have $T_n = tU_n$. Hence, by (7.6),

$$(7.9) \quad tS(dT_n/dt) = [(n-1)t^2 + S]T_n - nwt^2T_{n-1},$$

$$(T, t) = (X, x), (Y, y), (Z, z).$$

All 24 derivatives of the T_n ($T = U, V, W, X, Y, Z$) with respect to w, x, y, z are thus given by (7.3), (7.6)-(7.9), valid for all integers n .

8. *Differential Equations.* For the moment let accents denote derivatives with respect to w . Then, from (7.3), for T as there,

$$T_n' = nT_{n-1}, \quad T_n'' = n(n-1)T_{n-2}.$$

We have also

$$T_n - 2wT_{n-1} + NT_n = 0.$$

Eliminating T_{n-2}, T_{n-1} between these three equations we find

$$(8.1) \quad N(d^2T_n/dt^2) - 2(n-1)t(dT_n/dt) + n(n-1)T = 0,$$

$$(t = w; T = U, V, W, X, Y, Z).$$

Proceed similarly with (7.6), (7.8), which are of the same form. If for a moment we write

$$Q \equiv \frac{S}{t} \frac{dT_n}{dt}, \quad P \equiv \frac{S}{t} \frac{dQ}{dt},$$

the elimination of T_{n-2}, T_{n-1} (as in the preceding derivation) gives

$$\begin{vmatrix} P - (n-1)^2T_n & n(2n-3) & -n(n-1)w^2 \\ Q - (n-1)T_n & n & 0 \\ T_n & -2 & w^2 + S \end{vmatrix} = 0,$$

which is satisfied for all the (T, t) to which (7.6), (7.8) refer. Hence, upon reduction we find, after a short calculation,

$$(8.2) \quad tNS(d^2T_n/dt^2) - A(dT_n/dt) + (n-1)(n-2)t^3T_n = 0,$$

where

$$A \equiv NS + t^2[2(n-1)S - 3N],$$

$$(T, t) = (U, x), (U, y), (U, z),$$

$$(X, y), (X, z),$$

$$(Y, x), (Y, z),$$

$$(Z, x), (Z, y).$$

Similarly, from (7.9) we get

$$(8.3) \quad t^2 NS(d^2 T_n/dt^2) - tB(dT_n/dt) + CT_n = 0,$$

where

$$B \equiv 3NS + t^2 [2(n-1)S - 3N], \quad C \equiv B + (n-1)(n-2)t^4, \\ (T, t) = (X, x), \quad (Y, y), \quad (Z, z).$$

Finally, (7.7) gives in the same way,

$$(8.4) \quad tNS(d^2 T_n/dt^2) - D(dT_n/dt) + n(n-1)t^3 T_n = 0,$$

where

$$D \equiv NS + t^2 [2(n-1)S - N], \\ (T, t) = (V, x), \quad (V, y), \quad (V, z), \quad (W, x), \quad (W, y), \quad (W, z).$$

Thus (8.1)-(8.4) are the 24 possible differential equations for the six T_n ($T = U, V, W, X, Y, Z$) with respect to the four independent variables w, x, y, z . The equations hold for all integers n .

The self-adjoint equivalents of the foregoing equations are sufficiently simple to be noticed. Write $T_n \equiv \theta$. Then, corresponding to (8.1)-(8.4) respectively, we have their self-adjoint equivalents (8.11)-(8.41),

$$(8.j1) \quad (d/dt) [K_j(d\theta/dt)] - G_j\theta = 0 \quad (j=1, \dots, 4),$$

where

$$\begin{aligned} K_1 &\equiv N^{1-n}, & G_1 &\equiv n(1-n)N^{-n}; \\ K_2 &\equiv S^{3/2}N^{1-n}t^{-1}, & G_2 &\equiv (n-2)(1-n)tS^{1/2}N^{-n}; \\ K_3 &\equiv S^{3/2}N^{1-n}t^{-3}, & -G_3 &\equiv Ct^{-5}S^{1/2}N^{-n}; \\ K_4 &\equiv S^{1/2}N^{1-n}t^{-1}, & G_4 &\equiv n(1-n)tS^{-1/2}N^{-n}. \end{aligned}$$

In (8.j1) the same values (and only those) as in (8.j) of (T, t) are of course to be understood.

9. *Resolutions into factors; zeros.* Unless otherwise stated, let n be an integer > 0 . Referring to (4.9), we observe that $M_n(u)$ has only the n zeros $\mu_{n,j}$, all distinct, where

$$(9.1) \quad \mu_{n,j} \equiv \cos [j\pi/(n+1)] \quad (j=1, 2, \dots, n),$$

and that $L_n(u)$ has only the n zeros $\lambda_{n,j}$, all distinct where

$$(9.2) \quad \lambda_{n,j} \equiv \cos [(2j-1)\pi/2n] \quad (j=1, 2, \dots, n).$$

Hence, from (4.10), (4.11), noting the leading coefficients in $M_n(u)$, $L_n(u)$ as stated after (4.9), and taking the value of u as given by (4.9), (4.7), we have the following resolutions into factors,

$$(9.3) \quad T_{n+1} = 2^n t \prod_{j=1}^n (w - \mu_{n,j} N^{1/2}),$$

$$(T, t) = (U, 1), \quad (X, x), \quad (Y, y), \quad (Z, z);$$

and

$$(9.4) \quad T_n = 2^{n-1} t \prod_{j=1}^n (w - \lambda_{n,j} N^{1/2}),$$

$$(T, t) = (V, 2), \quad (W, 1).$$

If $m = -n < 0$, the resolutions of the T_m are written down from (9.3), (9.4) by (1.3), (1.12).

The above resolutions can readily be expressed rationally with respect to the independent variables. For, from (9.1), (9.2) we have

$$\mu_{n,j} = -\mu_{n,n-j+1}, \quad \lambda_{n,j} = -\lambda_{n,n-j+1} \quad (j = 1, \dots, n).$$

Hence, from (9.3), we have

$$(9.5) \quad T_{2n} = 2^{2n-2} t w \prod_{j=1}^{n-1} (w^2 - \mu_{2n-1,j}^2 N),$$

$$T_{2n+1} = 2^{2n} t \prod_{j=1}^n (w^2 - \mu_{2n,j}^2 N),$$

$$(T, t) = (U, 1), \quad (X, x), \quad (Y, y), \quad (Z, z);$$

and from (9.4),

$$(9.6) \quad T_{2n-1} = 2^{2n-2} t w \prod_{j=1}^{n-1} (w^2 - \lambda_{2n-1,j}^2 N),$$

$$T_{2n} = 2^{2n-1} t \prod_{j=1}^n (w^2 - \lambda_{2n,j}^2 N),$$

$$(T, t) = (V, 2), \quad (W, 1).$$

Each pair (9.5), (9.6) can be compressed into a single formula by using the greatest integer function $[n/2]$; there is but slight advantage in so doing.

Generalizing the concept of the zeros of a function of one variable to functions $P \equiv P(u_1, u_2, \dots, u_r)$ of $r > 1$ independent variables, real or complex, we define a zero of P to be a set $[u'_1, u'_2, \dots, u'_r]$ of values of u_1, u_2, \dots, u_r such that $P(u'_1, \dots, u'_r) = 0$. We proceed to determine all the zeros $[w', x', y', z']$ of $T_n(w, x, y, z)$ ($T = U, V, W, X, Y, Z$).

The unit sphere, in real or imaginary space of 3 dimensions,

$$(9.7) \quad \xi^2 + \eta^2 + \zeta^2 = 1,$$

is uniformized, among other ways, by the transformation to spherical coordinates,

$$(9.8) \quad \xi = \sin \theta \cos \phi, \quad \eta = \sin \theta \sin \phi, \quad \zeta = \theta,$$

which will suffice for our purpose. In (9.8), θ, ϕ are independent complex variables; the restriction, already imposed, to principal determinations of all irrations, may be repeated here for emphasis.

Let a, b, c, d be arbitrary numbers, real or complex, and write

$$(9.9) \quad \begin{aligned} \alpha_{n,j} &\equiv \tan(j\pi/2n), \quad \beta_{n,j} \equiv \tan[(j\pi/(2n+1))], \\ \gamma_{n,j} &\equiv \tan[(2j-1)\pi/(4n-2)], \quad \delta_{n,j} \equiv \tan[(2j-1)\pi/4n], \end{aligned}$$

where j is an integer. Then, with ξ, η, ζ as in (9.8), we see from (9.5) that T_{2n} , for $T = U, X, Y, Z$, has the zeros

$$(9.10) \quad [a, a\xi\alpha_{n,j}, a\eta\alpha_{n,j}, a\zeta\alpha_{n,j}] \quad (j = 1, 2, \dots, n-1);$$

T_{2n+1} , for $T = U, X, Y, Z$, has the zeros

$$(9.11) \quad [a, a\xi\beta_{n,j}, a\eta\beta_{n,j}, a\zeta\beta_{n,j}] \quad (j = 1, 2, \dots, n);$$

and from (9.6), T_{2n-1} , for $T = V, W$, has the zeros

$$(9.12) \quad [a, a\xi\gamma_{n,j}, a\eta\gamma_{n,j}, a\zeta\gamma_{n,j}] \quad (j = 1, 2, \dots, n-1);$$

T_{2n} , for $T = V, W$, has the zeros

$$(9.13) \quad [a, a\xi\delta_{n,j}, a\eta\delta_{n,j}, a\zeta\delta_{n,j}] \quad (j = 1, 2, \dots, n).$$

In addition to the zeros (9.10), T_{2n} has the zeros $(0, b, c, d)$, for $T = U, X, Y, Z$, and X_{2n}, Y_{2n}, Z_{2n} have also, respectively, the zeros

$$(9.14) \quad [a, 0, c, d], [a, b, 0, d], [a, b, c, 0].$$

In addition to the zeros (9.11), T_{2n+1} has the zeros (9.14) for $T = X, Y, Z$, respectively. In addition to the zeros (9.12), T_{2n-1} has the zeros $[0, b, c, d]$, for $T = V, W$. It is evident that all the zeros of T_n ($T = U, V, W, X, Y, Z$) have been stated in what precedes.

10. *Analogies with Weierstrass' σ, ζ, p .* Throughout this section, $N = 1$; and therefore $S = 1 - w^2$. Hence, by (4.2), (4.3), each of U_n, V_n is now a function of but one variable, w . We have now (10.1) $\phi_{2n} = \psi_{2n+1} = 0$, $\phi_{2n+1} = \psi_{2n} = (w^2 - 1)^n$, in place of (4.2). Thus the explicit forms of all

six T_n are known from (4.3), (4.4) with ϕ, ψ as in (10.1). Each of T_n ($T = X, Y, Z$) is now a function of precisely two independent variables, namely (w, t) ($t = x, y, z$), respectively, and W_n is a function of w alone. In a note to appear shortly elsewhere,* I have shown that Lucas' U_n, V_n present, up to a certain point, an isomorphism with the functions of Weierstrass, provided the Lucas functions have the parameter $(h, 1)$, where h is arbitrary. It follows from the above that the like holds for the six T_n . As further elaboration of this isomorphism would be merely a reproduction of the note cited, with a few obvious modifications, we may take it as accomplished by a reference to the note. As in the reference, the point of departure is (5.7), from which follows, in an obvious manner, a "relation of three terms" of precisely the same form as that for the sigma function.

11. *Generalizations.* One way in which the preceding admits of generalization has been pointed out in § 1. We note now that N is a norm in a linear algebra, and that the product of a sum of s squares by a sum of s squares is a sum of s squares only when $s = 2, 4, 8$. The case $s = 2$ gives the theory of the circular, hyperbolic, or Lucas functions; $s = 4$ has been treated above; the discussion of $s = 8$ will appear elsewhere. It may be stated here that the case $s = 8$ parallels $s = 4$ with but a few changes in definition and notation. Thus the 8 functions concerned are again Lucas functions; they are rationally resolvable into factors of degree ≤ 2 , etc., precisely as for $s = 4$. The case $s = 8$ includes $s = 4$. Again, in immediate sequence to the ideas of this paper, are those arising from the functions defined by powers of elements of an algebraic number field. I have elaborated the last in considerable detail. Last, the identity of Eisenstein (Crelle, 27, 105), is in an entirely different category; it does not belong to linear algebra.

* * * * *

The following considerations, not discussed in the body of this paper, may be of interest to those who would prefer the geometrical approach to the algebraic methods followed exclusively here. First, it will be clear that the fundamental formulas can be derived at once from simple considerations concerning rotations as interpreted in quaternions. This approach has been avoided for the following reason. The difference equation of the second order with constant coefficients which is satisfied by the functions and the polynomials is of course that whose associated quadratic equation is the characteristic equation satisfied by the general quaternion. Precisely the same

* *Bulletin of the American Mathematical Society.*

difference equation appears in the discussion of the polynomials and rational functions in eight variables which arise from a discussion of Cayley's eight-square imaginaries, which follows step by step the developments of this paper. In fact, the algebra of both cases could be treated in one presentation; they are abstractly identical. It would seem to follow then that the true geometrical interpretation of the quaternion case should apply equally well to that of the eight-square. Not being acquainted with any such interpretation of the eight-square imaginaries as rotations or other geometrical concepts, I have chosen the algebraic approach. It is significant also that the cases of four and eight squares exhaust the possibilities so far as polynomials and rational functions of the kind considered are concerned. In a geometrical approach this fact also should be accounted for.

Representations in the Form $xy + yz + zx$.

BY WALTER H. GAGE.

1. It has been shown by Bell* that the number of representations of an integer n in the form

$$(1) \quad xy + yz + zx$$

for which x, y, z are integers > 0 is $3[G(n) - 1]$ when and only when n is prime. In this paper we obtain by the same methods and formulae the results for any positive integer n for which x, y, z are integers ≥ 0 . By arithmetic means Mordell proved† that in this case the number of representations is $3G(n)$ provided we adopt the artificial convention that any solution in which one of the unknowns is zero be reckoned as $\frac{1}{2}$ instead of 1.

2. We first outline the notation and method and those results of Bell's paper which we need.

The $F(n), F_1(n)$ have their usual class number significance and $I(n), G(n)$ are respectively equal to $F(n) - 3F_1(n), F(n) + F_1(n)$. The symbol $\zeta(n)$ denotes the number of divisors of n ; $\zeta'(n)$ denotes the number of odd divisors of n ; and $\xi(n)$ is the number of divisors of n less than and of the opposite parity to their conjugates; $(-1 | n)$ is the Legendre symbol; and $\epsilon(n)$ equals 1 or 0 according as n is or is not the square of an integer. We define $\psi_1(m), \psi_2(m)$ by the properties: $\psi_1(m) = 0$ for $m \equiv 1 \pmod{4}$, but for $m \equiv 3 \pmod{4}$ $\psi_1(m)$ is defined to be the number of divisors of m less than their conjugates; $\psi_2(m)$ has the same properties if the congruences be interchanged. Throughout the paper m denotes an odd integer > 0 .

Let $f(x)$ be an even function of x so that $f(x) = f(-x)$ which is entirely arbitrary except that $f(x)$ takes a single definite value for each x an integer ≥ 0 . If

$$(2) \quad a_0 + \sum_{i=1}^r a_i \cos n_i x = 0$$

is an identity in x , the a_i, n_i being integers, then

* *Tôhoku Mathematical Journal*, Vol. 19, 1921.

† *American Journal of Mathematics*, Vol. 45, 1923.

$$(3) \quad a_0 + \sum_{i=1}^r a_i f(n_i) = 0.$$

We may take $f(x) = \phi(x)$ where

$$(4) \quad \phi(0) = 1, \quad \phi(x) = 0, \quad |x| > 0,$$

whence we get

$$(5) \quad a_0 + \sum_{i=1}^r a_i \phi(n_i) = 0.$$

From the theta-function identities

$$\begin{aligned} \vartheta_0 \frac{\vartheta_2(x)\vartheta_3(x)}{\vartheta_1(x)} \times \vartheta_0\vartheta_3 \frac{\vartheta_2(x)}{\vartheta_1(x)} &= \vartheta_0^2\vartheta_3 \frac{\vartheta_2^2(x)\vartheta_3(x)}{\vartheta_1^2(x)}, \\ \vartheta_2 \frac{\vartheta_0(x)\vartheta_3(x)}{\vartheta_1(x)} \times \vartheta_2\vartheta_3 \frac{\vartheta_0(x)}{\vartheta_1(x)} &= \vartheta_2^2\vartheta_3 \frac{\vartheta_0^2(x)\vartheta_3(x)}{\vartheta_1^2(x)}, \end{aligned}$$

by expanding in powers of q , equating the coefficients of like powers of q , and applying the above theorem with $\phi(x)$ as in (4) we obtain the following.

I. Let $N_J(m)$ be the number of representations of m in the form (1) for which

$$0 < y, z \equiv 1 \pmod{2}; y < z; 0 < x \equiv J \pmod{2}.$$

Then

$$N_0(m) - N_1(m) = I(m) - \frac{1}{2}[1 + (-1 \mid m)]\xi(n) + \epsilon(m)(-1 \mid m)/2.$$

II. Let $M(n)$, n an arbitrary integer, denote the number of representations of n in the form (1) for which

$$0 < x, y, z; x + y \equiv 1 \pmod{2}.$$

Then

$$M(n) = 2F(n) - \zeta'(n) - \frac{1}{2}[1 + (-1)^n]\xi(n).$$

The number of representations of n in the form (1) when two of $x, y, z > 0$ are equal is obviously deducible from the number of positive integral solutions for n fixed of $n = u^2 + 2uv$, which is the number of divisors of n of the same parity as but less than their conjugates. The number of representations of n in the form (1) with one of x, y, z zero depends upon the number of positive integral solutions of $n = uv$ and is $3\xi(n)$.

3. Let n be odd, and replace it by m in II. When n is odd x, y, z

must be either all odd and $m \equiv 3 \pmod{4}$, or two odd and one even with $m \equiv 1 \pmod{4}$. In the first case the number $N_1'(m)$ of representations where $x, y, z \geq 0$ is

$$(6) \quad N_1'(m) = N_1(m) + \psi_1(m).$$

In the second case the number $N_0'(m)$ of representations is

$$(7) \quad N_0'(m) = 3F(m) - (3/2)\xi(m) + 3\zeta(m) = 3[F(m) + 1/2\zeta(m)].$$

From the definitions of $N_0'(m)$, $N_0(m)$ we have

$$(8) \quad 1/3 N_0'(m) - \psi_2(m) - \zeta(m) = N_0(m).$$

Substitute in (8) the value of $N_0'(m)$ obtained from (7) and put the resulting value of $N_0(m)$ in I, thus determining $N_1(m)$. Hence from (6) we get $N_1'(m)$. The total number $N(m)$ of representations for which $x, y, z \geq 0$ is the sum of $N_0'(m)$ and $N_1'(m)$, which are now known. Using the identities $\zeta(m) = 2\psi_1(m) = 2\psi_2(m) + \epsilon(m)$ we find

$$(9) \quad N(m) = 3[G(m) + 1/2\zeta(m)].$$

Mordell's result follows when, adopting his convention, we subtract $(3/2)\zeta(m)$, which is half the number of representations of m for which one of x, y, z is zero.

Next, let $n = 2^\alpha m$, $\alpha > 0$, m odd. It is easily seen that x, y, z must be all even, or only two even. Let the total number of representations in these cases, for which $x, y, z \geq 0$, be $N_0(n)$, $N_1(n)$ respectively. In the latter case it follows from § 2 that for one of x, y, z zero the number of representations is $6\zeta(m)$, since the factors u, v must be of opposite parities. Hence from II

$$(10) \quad N_1(2^\alpha m) = 3F(2^\alpha m) + 3\zeta(m).$$

Suppose now that x, y, z are all even. We then divide out the factor 4 and consider the number of representations of $2^{\alpha-2}m$ in the form (1). Some of the solutions are again given by (10) where 2^α is replaced by $2^{\alpha-2}$. To find the remaining representations again divide out a factor 4 and proceed as above. Continue this process until $2m$ or m is reached, according as α is odd or even. The number of representations of $2m$ is $N_1(2m)$, of m it is $3[G(m) + 1/2\zeta(m)]$. Hence if the total number of representations is $N(n)$ we have finally

$$(11) \quad \alpha \text{ odd: } N(2^\alpha m) = 3[F(2^\alpha m) + F(2^{\alpha-2}m) + \cdots + F(2m) + (\alpha + 1/2)\zeta(m)],$$

α even: $N(2^am) = 3[F(2^am) + F(2^{a-2}m) + \cdots + G(m) + (\alpha + 1/2)\zeta(m)]$.

Both reduce by the class number relations $F(4n) = 2F(n)$, $E(4n) = E(n)$ to

$$(12) \quad N(2^am) = 3[G(2^am) + (\alpha + 1/2)\zeta(m)].$$

The result in this form holds true for $\alpha \geq 0$ and hence for all positive integers. Adopting Mordell's convention we get his result immediately.

VICTORIA COLLEGE,
VICTORIA, B. C.

The Resultant of Two Power Series in Two Variables.

By L. L. DINES.

For a system of r power series

$$f_i(x_1, x_2, \dots, x_r) \quad (i = 1, 2, \dots, r),$$

each without constant term and converging in some neighborhood of the origin there is in general a neighborhood of the origin in which the system of simultaneous equations $f_1 = 0, f_2 = 0, \dots, f_r = 0$ admits no solution except the trivial one $x_1 = x_2 = \dots = x_r = 0$.*

In order that this system of equations shall admit a nontrivial solution in every neighborhood of the origin, it is obvious then that the coefficients of the series f_i must satisfy special conditions. In the very particular case in which each series is merely a homogeneous polynomial, the condition is known to be the vanishing of a well defined rational integral function of the coefficients—the resultant of the homogeneous polynomials.†

The generalization to power series has not been made. A direct method which suggests itself for obtaining the desired condition is to eliminate $r - 1$ of the variables from the r equations. If we assume the series f_i to be formal power series (with literal coefficients) of positive order, this elimination can be effected (theoretically at least) by either of two methods: (1) the general elimination algorithm for power series as developed by Kistler,‡ (2) the beautiful elimination theorem due to Bliss.§

Each of these methods leads to a power series in a single variable, the vanishing of whose coefficients appears to be the condition sought. The difficulty lies in the fact that the coefficients of this resultant series, though rational in the coefficients of the given series, are not *integral* functions. Hence the resultant series becomes meaningless if the literal coefficients of

* Cf. Osgood, *Lehrbuch der Funktionentheorie II*, p. 104: "The second theorem of Weierstrass."

† Cf. J. König, *Einleitung in die allgemeine Theorie der algebraischen Größen*, p. 295.

‡ *Über Funktionen von mehreren komplexen Veränderlichen*, Dissertation, Göttingen (1905).

§ A generalization of Weierstrass' Preparation Theorem for a power series in several variables, *Transactions of the American Mathematical Society*, Vol. 13, p. 133.

the given series be replaced by numerical values which cause denominators in the resultant series to vanish.

This difficulty can be overcome at least in the simplest case $r=2$ and it is the object of the present paper to develop a theory of the resultant for that case.

In § 12 an application is made to the theory of implicit functions.

1. *Statement of some properties of the resultant.* We shall consider two formal power series of orders m and n respectively. These we shall write in the form

$$\begin{aligned} (1) \quad f(x, y) &\equiv f_m + f_{m+1} + \cdots \\ g(x, y) &\equiv g_n + g_{n+1} + \cdots \end{aligned}$$

where f_j and g_j are homogeneous polynomials (forms) of degree j in the two variables x, y , with literal coefficients. The coefficients in f will be denoted generically by a and those in g by b . Relative to these two series we shall establish the following:

THEOREM: *There exists an infinite sequence*

$$r \equiv (R_{mn}, R_{mn+1}, R_{mn+2}, \cdots)$$

of well defined functions of the coefficients a and b , having the following properties:

(a) If the coefficients a and b in the formal series be assigned numerical values such that the two series retain their respective orders m and n and converge in a neighborhood of the origin, then the two equations $f=0, g=0$ will admit a common solution $(x, y) \neq (0, 0)$ in every neighborhood of the origin if and only if the sequence r vanishes identically.

(b) Each element R_{mn+k} of the sequence r is a rational function of the coefficients a and b and of four auxiliary symbols $u_{11}, u_{12}, u_{21}, u_{22}$ (generically u). The only denominators occurring in these rational functions are products of powers of the two forms $f_m(u_{11}, u_{21})$ and $g_n(u_{11}, u_{21})$.

(c) The first element R_{mn} of the sequence is the product of two factors of which one is the resultant of the two leading forms f_m, g_n , and the other is $|u_{ij}|^{mn}$.

(d) Each element R_{mn+k} is rationally homogeneous* in the coefficients

* The terms *rationally homogeneous* and *rationally isobaric* are to be understood in the following sense. A rational function of a set of symbols is said to be rationally homogeneous if it is a quotient of two homogeneous polynomials. Its degree is the

a and b and the auxilliary symbols u . Its degree is n in a , m in b , and $2mn + k$ in u .

(e) Each element R_{mn+k} is rationally isobaric* of weight $2mn + k$ in the combined set of coefficients a, b , the weight of a coefficient being the degree of the form f_j or g_j in which it occurs.

The sequence r will be called the *resultant* of the two power series f and g . Its derivation is described in § 2. Property (a) is proven in § 3, and the remaining properties in later sections after a prerequisite formal study of the important Weierstrass Preparation Theorem.

2. *Derivation of the sequence r .* To the variables x, y of the two formal series (1) we apply the linear homogeneous transformation

$$(2) \quad \begin{aligned} x &= u_{11}\xi + u_{12}\eta \\ y &= u_{21}\xi + u_{22}\eta \end{aligned}$$

the coefficients u being entirely arbitrary except for the restriction $|u_{ij}| \neq 0$; and consider the resulting series in the new variables ξ, η . These may be written

$$(3) \quad \begin{aligned} \phi(\xi, \eta) &\equiv \phi_m + \phi_{m+1} + \cdots \\ \psi(\xi, \eta) &\equiv \psi_n + \psi_{n+1} + \cdots \end{aligned}$$

where ϕ_j and ψ_j denote homogeneous polynomials in ξ, η , with coefficients homogeneous of degree j in u . The coefficients in ϕ and ψ will be denoted generically by α and β respectively. In particular, the coefficient of ξ^m in ϕ_m and the coefficient of ξ^n in ψ_n will be denoted by α_{m0} and β_{n0} respectively, and they are easily seen to have the forms

$$(4) \quad \alpha_{m0} = f_m(u_{11}, u_{21}), \quad \beta_{n0} = g_n(u_{11}, u_{21}).$$

To the two series $\phi(\xi, \eta), \psi(\xi, \eta)$ we now apply formally the Preparation Theorem of Weierstrass.* We thus obtain two formal identities

$$(5) \quad \mu\phi = p, \quad \nu\psi = q,$$

degree of the numerator minus the degree of the denominator. If with each symbol of the set there is associated, by means of any definite law, a positive integer called the weight of that symbol, then the weight of any product of symbols of the set is defined to be the sum of the weights of the factors. A polynomial in symbols of the set is isobaric of weight w if each of its terms is of weight w . A rational function of the symbols is rationally isobaric if it is a quotient of isobaric polynomials. Its weight is the weight of the numerator minus the weight of the denominator.

* Cf. Bliss, *Princeton Colloquium Lectures*, p. 50.

where the multipliers μ and ν are power series in ξ, η with constant term unity, and p and q are polynomials in ξ of degrees m and n respectively. These polynomials may be written

$$(6) \quad \begin{aligned} p &\equiv \alpha_0 \xi^m + \alpha_1 \xi^{m-1} + \cdots + \alpha_m \\ q &\equiv \beta_0 \xi^n + \beta_1 \xi^{n-1} + \cdots + \beta_n \end{aligned}$$

the coefficients α_0 and β_0 being equal to α_{m0} and β_{n0} respectively, while the remaining coefficients α_j and β_j are power series in η without constant terms. The multipliers and polynomials are uniquely determined by these conditions.

We next form the algebraic resultant of the two polynomials p, q . This resultant is a well defined rational integral function of the coefficients of the two polynomials and hence is expressible as a power series in η , which we shall call the resultant series and denote by $r(\eta)$. *The sequence of coefficients of the series $r(\eta)$ is the sequence*

$$r \equiv (R_{mn}, R_{mn+1}, \cdots)$$

which we have called the resultant of the two series f and g .

3. *Proof of property (a), and an equivalent statement.* Suppose that the literal coefficients of the series $f(x, y)$ and $g(x, y)$ are assigned numerical values such that the two series preserve their respective orders m and n , and converge in a neighborhood of the origin.

From the nature of the transformation (2) it follows that for every set of finite values which may be assigned to the symbols u , the two series $\phi(\xi, \eta)$ and $\psi(\xi, \eta)$ will converge in some neighborhood of $(\xi, \eta) = (0, 0)$. Furthermore, since $|u_{ij}| \neq 0$, the existence of a common solution $(x, y) \neq (0, 0)$ of the two equations $f = 0, g = 0$ in every neighborhood of the origin is equivalent to the existence of a common solution $(\xi, \eta) \neq (0, 0)$ of the two equations $\phi = 0, \psi = 0$ in every neighborhood of the origin.

If the root-systems of the homogeneous equations $f_m = 0, g_n = 0$ be excluded from the possible values of the auxiliary symbols u_{11}, u_{21} , then the coefficients (4) will be different from zero, and the identities (5) will be valid in a neighborhood of $(\xi, \eta) = (0, 0)$. Since μ and ν have constant terms different from zero, the existence of a common non-trivial solution of $\phi = 0, \psi = 0$ in every neighborhood of the origin is equivalent to the existence of a like solution of $p = 0, q = 0$ in every neighborhood of the origin. But the criterion for the latter condition is the identical vanishing of the resultant series $r(\eta)$, that is the identical vanishing of the sequence r . This establishes property (a).

Equivalent to property (a) is the following:

Property (a'): Under the same hypotheses as in (a), the vanishing of the resultant r is necessary and sufficient for the existence of a non-trivial common factor of the series $f(x, y)$ and $g(x, y)$; in other words, for the existence of identities

$$(7) \quad f(x, y) \equiv f'(x, y)d(x, y), \quad g(x, y) \equiv g'(x, y)d(x, y),$$

where f' , g' , and d are power series and d has no constant term.

First, suppose the resultant r vanishes. Then the two polynomials (6) admit a common factor polynomial in ξ .* We may express this fact by identities

$$(8) \quad p \equiv p'\delta, \quad q \equiv q'\delta$$

where p' , q' , and δ are polynomials of the same type as p and q .

Substituting (8) in (5), we obtain identities which may be written

$$(9) \quad \phi \equiv p''\delta, \quad q \equiv q''\delta,$$

where $p'' = p'/\mu$ and $q'' = q'/\nu$.

Now assigning any non-singular numerical values to the auxiliary symbols u , and applying the inverse of the substitution (2) to (9), we obtain identities of the specified form (7).

Conversely, suppose the given series satisfy identities of the form (7). Then due to the common factor $d(x, y)$, the equations $f = 0$, $g = 0$ admit a common non-trivial solution in every neighborhood of the origin, and hence the resultant must vanish by property (a).

4. *The discriminant of a single series in two variables.* Consider the single formal power series

$$f(x, y) = f_m + f_{m+1} + \dots,$$

and the series

$$\phi(\xi, \eta) = \phi_m + \phi_{m+1} + \dots$$

obtained by the substitution (2).

To the related polynomial

$$p = \alpha_0 \xi^m + \alpha_1 \xi^{m-1} + \dots + \alpha_m,$$

obtained by the Weierstrass Preparation Theorem we may apply the formal differentiation process, and obtain a polynomial

* Cf. Osgood, *loc. cit.*, p. 84.

$$p_{\xi} = m \alpha_0 \xi^{m-1} + (m-1) \alpha_1 \xi^{m-2} + \cdots + \alpha_{m-1}.$$

The algebraic resultant of the two polynomials p and p_{ξ} is a series which we may call the discriminant series of the series $f(x, y)$ and denote by $d(\eta)$. The sequence of coefficients constitutes the discriminant d of the series $f(x, y)$.

If the literal coefficients in $f(x, y)$ be assigned numerical values such that the series converges and retains its order, then the identical vanishing of the discriminant d is necessary and sufficient for the existence of a repeated polynomial factor of the polynomial p ,* and hence for the existence of a repeated non-trivial series factor of the series $f(x, y)$.

5. *Formal properties of the Weierstrass Preparation Identity.* The properties (b)–(e) of § 1 are formal properties of the resultant r which depend upon certain formal properties inherent in the Weierstrass Preparation Identity. These latter properties will be established in the present section.

According to the Weierstrass theorem, a formal series $\phi(\xi, \eta)$ of order m satisfies an identity

$$(10) \quad \mu\phi = p$$

where μ and p are power series in ξ and η with certain special properties; namely, μ has a constant term unity while p involves no powers of ξ higher than the $(m-1)$ th except the single term $\alpha_{m0}\xi^m$.

Adopting as generic notations for coefficients in ϕ , μ and p the respective letters α , γ , ρ , and taking account of the above mentioned properties, we may assume the following expressions:

$$\begin{aligned} \phi &\equiv \phi_m + \phi_{m+1} + \cdots \\ \mu &\equiv 1 - \mu_1 - \mu_2 - \cdots \\ p &\equiv p_m + p_{m+1} + \cdots \\ \phi_j &\equiv \alpha_{j0}\xi^j + \alpha_{j-1,1}\xi^{j-1}\eta + \cdots + \alpha_{1,j-1}\xi\eta^{j-1} + \alpha_{0j}\eta^j \quad (j \geq m) \\ \mu_k &\equiv \gamma_{k0}\xi^k + \gamma_{k-1,1}\xi^{k-1}\eta + \cdots + \gamma_{1,k-1}\xi\eta^{k-1} + \gamma_{0k}\eta^k \quad (k \geq 1) \\ p_m &\equiv \rho_{m0}\xi^m + \rho_{m-1,1}\xi^{m-1}\eta + \cdots + \rho_{1,m-1}\xi\eta^{m-1} + \rho_{0m}\eta^m \\ p_j &\equiv \rho_{m-1,j-m+1}\xi^{m-1}\eta^{j-m+1} + \cdots + \rho_{1,j-1}\xi\eta^{j-1} + \rho_{0j}\eta^j \quad (j > m) \end{aligned}$$

Substituting the expressions for ϕ , μ , and p in (10), and equating forms of the same degree on the two sides of the resulting identity, we obtain a system of identities which may be written

$$(11)_0 \quad p_m = \phi_m$$

$$(11)_1 \quad \phi_m \mu_1 + p_{m+1} = \phi_{m+1}$$

* Cf. Osgood, *loc. cit.*, p. 88.

$$(11)_k \quad \phi_m \mu_k + p_{m+k} = \phi_{m+k} - \sum_{i=1}^{k-1} \mu_i \phi_{m+k-i} \quad (k = 2, 3, \dots).$$

It is almost obvious that these identities determine the coefficients of μ_k and p_{m+k} recursively, but it will be useful to note some of the details of the determination.

The identity $(11)_0$ determines the coefficients of p_m , viz.

$$(12) \quad p_{m0} = \alpha_{m0}, \dots, p_{m-j,j} = \alpha_{m-j,j}, \dots, p_{0m} = \alpha_{0m}.$$

The identity $(11)_1$ determines the coefficients of p_{m+1} and μ_1 . For substituting the explicit forms in $(11)_1$, and equating coefficients of like power products of ξ, η , we obtain the system of equations

$$(13) \quad \begin{array}{rcl} \alpha_{m0}\gamma_{10} & & = \alpha_{m+1,0} \\ \alpha_{m-1,1}\gamma_{10} + \alpha_{m0}\gamma_{01} & & = \alpha_{m1} \\ \alpha_{m-2,2}\gamma_{10} + \alpha_{m-1,1}\gamma_{01} + \rho_{m-1,2} & & = \alpha_{m-1,2} \\ \alpha_{m-3}\gamma_{10} + \alpha_{m-2,2}\gamma_{01} & + \rho_{m-2,3} & = \alpha_{m-2,3} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \alpha_{0m}\gamma_{10} + \alpha_{1,m-1}\gamma_{01} & + \rho_{1m} & = \alpha_{1m} \\ & \alpha_{0m}\gamma_{01} & + \rho_{0,m+1} = \alpha_{0,m+1}. \end{array}$$

This system of $m+2$ linear equations suffices to determine the $m+2$ coefficients $\gamma_{10}, \gamma_{01}, \rho_{m-1,2}, \dots, \rho_{0,m+1}$, the determinant of the system being equal to α_{m0}^2 .

In a similar manner, each identity $(11)_k$ leads to a system of linear equations which uniquely determines the coefficients of μ_k and p_{m+k} in terms of the coefficients of μ_j and p_{m+j} ($j < k$).

We are now in a position to note certain properties of homogeneity and isobaricity of the coefficients γ and ρ . The terms *rationaly homogeneous* and *rationaly isobaric* have been defined in a footnote under § 1. Defining the *weight* of a coefficient α to be the degree k of the form ϕ_k in which it appears, we shall use the notation

$$[\alpha]_{\omega}^d$$

to denote generically a function of the coefficients α which is rationaly homogeneous of degree d and rationaly isobaric of weight ω in these coefficients. Then we shall see that

- (i) Each coefficient γ in μ_k has the character $[\alpha]_k^0$.
- (ii) Each coefficient ρ in ρ_k has the character $[\alpha]_k^1$.
- (iii) The only denominators occurring in the coefficients γ and ρ are powers of α_{m0} .

These properties are obviously true for the coefficients of p_m in (12). They can be verified easily for the coefficients of μ_1 and p_{m+1} from the form of the system of equations (13). Their general validity follows by mathematical induction from a consideration of the system of linear equations arising from the identity $(11)_k$. The matrix of coefficients of this system will be similar to that in (3). There are $k+1$ columns headed by α_{m0} , and there are m elements 1 in the principal diagonal. The value of the determinant is therefore α_{m0}^{k+1} . The right side of each equation has the character $[\alpha]^{1_{m+k}}$ under the assumption that the properties hold for the coefficients of μ_i ($i < k$). The Cramer rule for the solution of a system of equations then makes the properties (i), (ii), and (iii) apparent.

6. *Formal properties of $p(\xi, \eta)$ as a polynomial in ξ .* The series $p(\xi, \eta)$ appearing in the identity (10), when considered as a polynomial in ξ , may be written

$$p(\xi, \eta) \equiv \alpha_0 \xi^m + \alpha_1 \xi^{m-1} + \cdots + \alpha_m.$$

Since, as we have seen, $p(\xi, \eta)$ is a series of order m in ξ, η , the coefficient α_j is a series of order j in η :

$$(14) \quad \alpha_j(\eta) \equiv \rho_{m-j, j} \eta^j + \rho_{m-j, j+1} \eta^{j+1} + \rho_{m-j, j+2} \eta^{j+2} + \cdots$$

Making use of property (ii) of the preceding section, we see that each series $\alpha_j(\eta)$ has as coefficients a sequence of functions which have the respective characters

$$[\alpha]^{1_m}, [\alpha]^{1_{m+1}}, [\alpha]^{1_{m+2}}, \cdots$$

7. *Formal properties of the resultant r .* The resultant r of two series $f(x, y)$ and $g(x, y)$ has been defined in § 2 as the sequence of coefficients of a certain power series which represents the algebraic resultant of two polynomials

$$\begin{aligned} p &\equiv \alpha_0 \xi^m + \alpha_1 \xi^{m-1} + \cdots + \alpha_m \\ q &\equiv \beta_0 \xi^n + \beta_1 \xi^{n-1} + \cdots + \beta_n. \end{aligned}$$

The coefficients α_0 and β_0 are equal to α_{m0} and β_{n0} respectively. For $j > 0$, the coefficients α_j and β_j are power series in η , the formal characters of which are indicated by

$$(15) \quad \begin{aligned} \alpha_j: & \quad [\alpha]^{1_m} \eta^j + [\alpha]^{1_{m+1}} \eta^{j+1} + [\alpha]^{1_{m+2}} \eta^{j+2} + \cdots \\ \beta_j: & \quad [\beta]^{1_n} \eta^j + [\beta]^{1_{n+1}} \eta^{j+1} + [\beta]^{1_{n+2}} \eta^{j+2} + \cdots \end{aligned}$$

The only denominators occurring in the coefficients of α_j are powers of α_{m0} , and the only denominators in the coefficients of β_j are powers of β_{n0} .

Now the formal properties of algebraic resultants are well known. According to the algebraic theory, the resultant of the two polynomials p, q , is a rational integral function of the coefficients, and has the following additional properties: (1) It is homogeneous of degree n in the coefficients of p , and homogeneous of degree m in the coefficients of q ; (2) it is isobaric of weight mn in the combined set of coefficients, the weight of α_j and β_j being defined to be j .

From the second of these properties and the expressions (15) for α_j and β_j it follows that the resultant of p and q is a power series of order mn in η . We write it

$$r(\eta) \equiv R_{mn}\eta^{mn} + R_{mn+1}\eta^{mn+1} + \dots$$

From the properties (1) and (2) and the expressions for α_j and β_j we obtain also the following formal properties relative to the coefficients R_{mn+k} . (The designations (b)-(e) are used to correlate these properties with the properties similarly designated in § 1.)

(b) Each coefficient R_{mn+k} is a rational function of the coefficients α and β , and the only denominators appearing are products of powers of α_{m0} and β_{n0} .

(c) The first coefficient R_{mn} is the algebraic resultant of the leading polynomials $\phi_m(\xi, \eta)$ and $\psi_n(\xi, \eta)$ of the series ϕ and ψ . This follows from the fact that the leading coefficient of the series $\alpha_j(\eta)$ is $\alpha_{m-j,j}$ (in view of (14) and (12)), and the leading coefficient of the series $\beta_j(\eta)$ is $\beta_{m-j,j}$.

(d) Each coefficient R_{mn+k} is rationally homogeneous of degree n in the coefficients α and rationally homogeneous of degree m in the coefficients β .

(e) Each coefficient R_{mn+k} is rationally isobaric of weight $2mn + k$ in the combined set of coefficients α, β .

8. *Proof of properties (b) . . . (e) of § 1.*

In § 2, the given series $f(x, y)$ and $g(x, y)$ were replaced by two series $\phi(\xi, \eta)$ and $\psi(\xi, \eta)$ by means of the transformation

$$\begin{aligned} (2) \quad x &= u_{11}\xi + u_{12}\eta \\ y &= u_{21}\xi + u_{22}\eta \end{aligned}$$

In § 7 formal properties of the resultant r have been obtained in terms of the coefficients α, β of the series ϕ and ψ . It is easy to translate these properties into terms of the coefficients a and b .

From the nature of the transformation (2) it follows that each coefficient

α of the polynomial $\phi_k(\xi, \eta)$ is homogeneous in the coefficients a of the polynomial $f_k(x, y)$ and in the auxiliary symbols u . It is linear in the former and of degree k in the latter. An analogous statement can be made relative to each coefficient β in $\psi_k(\xi, \eta)$.

In view of these facts the properties (b)-(e) of § 1 follow from the similarly designated properties in § 7. In verifying (b) reference may be made to formulas (4). The factorization of R_{mn} indicated in (c) is a consequence of the well known invariant property of algebraic resultants.*

9. *The Product Theorem for resultants of power series.* Let $f'(x, y)$ and $g'(x, y)$ be a pair of formal power series of orders m' and n' with resultant

$$r' \equiv (R'_{m'n'}, R'_{m'n'+1}, \dots),$$

and let $f''(x, y)$ and $g''(x, y)$ be a second pair of formal power series of orders m'' and n'' with resultant

$$r'' \equiv (R''_{m''n''}, R''_{m''n''+1}, \dots),$$

Then the resultant of the two product series

$$f \equiv f'f'' \text{ and } g \equiv g'g''$$

is

$$r \equiv (R_{mn}, R_{mn+1}, \dots)$$

where

$$R_{mn} = R'_{m'n''} R''_{m''n''}$$

$$R_{mn+k} = \sum_{i=0}^k R'_{m'n'+i} R''_{m''n''+k-i}, \quad (k = 1, 2, 3, \dots).$$

To prove the theorem, we apply the substitution (2) to each of the four factor series and to the two product series, obtaining six series

$$\phi'(\xi, \eta), \psi'(\xi, \eta), \phi''(\xi, \eta), \psi''(\xi, \eta), \phi(\xi, \eta), \psi(\xi, \eta),$$

satisfying the two identities

$$\phi'(\xi, \eta) \cdot \phi''(\xi, \eta) = \phi(\xi, \eta); \quad \psi'(\xi, \eta) \psi''(\xi, \eta) = \psi(\xi, \eta).$$

To each of the series ϕ' , ϕ'' , ψ' , ψ'' , ϕ , ψ , we apply the Weierstrass Preparation Theorem, obtaining six identities

$$\mu' \phi' = p', \quad \mu'' \phi'' = p'', \quad \nu' \psi' = q', \quad \nu'' \psi'' = q'', \quad \mu \phi = p, \quad \nu \psi = q.$$

Equating the products of corresponding sides of the first two of these identities we obtain

$$\mu' \mu'' \phi' \phi'' = p' p''.$$

* Cf. J. König, *loc. cit.*, pp. 293-4.

Comparing this with the identity

$$\mu \phi = p$$

we conclude that

$$(16) \quad p = p'p''.$$

For $\phi'\phi'' = \phi$, and $\mu'\mu''$ is a series with constant term unity. Hence the uniqueness of the Weierstrass identity necessitates (16). Similarly we see that

$$(17) \quad q = q'q''.$$

The three resultants r' , r'' , r , are the sequences of coefficients of three resultant series $r'(\eta)$, $r''(\eta)$, $r(\eta)$, which are by definition the algebraic resultants of the three pairs of polynomials (p', q') , (p'', q'') , and (p, q) respectively.

Hence it follows from (16), (17), and the product theorem* for algebraic resultants that $r(\eta) = r'(\eta) \cdot r''(\eta)$.

The resulting relations between the coefficients of the three series yield our theorem.

10. *The effect of a linear transformation of variables.* If two series $f(x, y)$ and $g(x, y)$ with resultant r be subjected to a linear homogeneous transformation of variables

$$(18) \quad \begin{aligned} x &= c_{11}x' + c_{12}y' \\ y &= c_{21}x' + c_{22}y' \end{aligned} \quad |c_{ij}| \neq 0,$$

the resulting pair of series $f'(x', y')$, $g'(x', y')$ will have a resultant which we may denote by r' . The relation between the two resultants r and r' is easily determined.

The resultant r is obtained by a definite procedure from two series $\phi(\xi, \eta)$, $\psi(\xi, \eta)$ which are the transforms of $f(x, y)$ and $g(x, y)$ under the transformation

$$(2) \quad \begin{aligned} x &= u_{11}\xi + u_{12}\eta \\ y &= u_{21}\xi + u_{22}\eta. \end{aligned}$$

The resultant r' is obtained by the same procedure from two series $\phi'(\xi, \eta)$, $\psi'(\xi, \eta)$ which are the transforms of $f'(x', y')$ and $g'(x', y')$ under the transformation

$$(19) \quad \begin{aligned} x' &= u_{11}\xi + u_{12}\eta \\ y' &= u_{21}\xi + u_{22}\eta. \end{aligned}$$

* Cf. J. König, *loc. cit.*, p. 288.

But the two series $\phi'(\xi, \eta)$, $\psi'(\xi, \eta)$ can of course be obtained directly from $f(x, y)$, $g(x, y)$ by a single transformation which is the product of (18) and (19), that is by the transformation

$$(2') \quad \begin{aligned} x &= u'_{11}\xi + u'_{12}\eta \\ y &= u'_{21}\xi + u'_{22}\eta \end{aligned}$$

where the coefficients u' are given by the matrix equation

$$(20) \quad \begin{pmatrix} u'_{11} & u'_{12} \\ u'_{21} & u'_{22} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}.$$

Since the series $f(x, y)$, $g(x, y)$ are transformed by (2) into $\phi(\xi, \eta)$, $\psi(\xi, \eta)$ and by (2') into $\phi'(\xi, \eta)$, $\Psi'(\xi, \eta)$, it follows that the resultants r and r' differ only in the auxiliary symbols. More explicitly, r' may be obtained from r by a transformation of auxiliary symbols indicated by the matrix identity (20).

If in particular we take for (18) the very simple transformation $x = y'$, $y = x'$, we note that the only effect upon the resultant is an interchange of auxiliary symbols.

11. *The order of the resultant of two series with numerical coefficients.* For two formal series $f(x, y)$, $g(x, y)$ of orders m and n , the resultant is, as we have seen, a sequence

$$r \equiv (R_{mn}, R_{mn+1}, \dots).$$

If numerical values be assigned to the coefficients of f and g , this sequence may vanish identically, or some of its terms may vanish while others do not. If the first non-vanishing element of the sequence r is R_{mn+k} , the resultant of the two series f and g will be said to be of order $mn + k$.

From the developments of the preceding section it follows that the order of the resultant is invariant under transformations of the form (18). This invariant may be interpreted geometrically with reference to the intersection of the two surfaces $z = f(x, y)$ and $z = g(x, y)$ in the neighborhood of the origin. Of somewhat more general significance is the application indicated in the next section.

12. *Application of the resultant to the theory of implicit functions.* A fundamental question in the theory of implicit functions has to do with the number and character of the solutions of a system of r equations

$$(21) \quad \begin{aligned} F_i(x_1, x_2, \dots, x_r; z_1, z_2, \dots, z_s) &= 0, \\ (i &= 1, 2, \dots, r), \end{aligned}$$

for r dependent variables x_1, \dots, x_r in the neighborhood of a point at which the functions F_i vanish.

Suppose the functions F_i are power series in the indicated variables, convergent in a neighborhood of the origin, at which point all of the series vanish. Suppose further that the following condition is satisfied.

Condition O. The r equations

$$F_i(x_1, x_2, \dots, x_r; 0, \dots, 0) \quad (i = 1, 2, \dots, r)$$

admit no common non-trivial solution in the neighborhood of the origin.

Then a characterization of the solutions of the system (21) in the neighborhood of the origin may be made by means of a related polynomial

$$E(\eta; z_1, z_2, \dots, z_s) \equiv \eta^N + e_1\eta^{N-1} + \dots + e_N$$

where η is a suitably chosen linear combination of the variables x_1, x_2, \dots, x_r , and the coefficients e_j are power series in z_1, z_2, \dots, z_s vanishing with these variables.*

The relationship of this polynomial $E(\eta)$ to the system of equations (21) is such that it may properly be called an *eliminating polynomial* for the dependent variables x , and its degree N may be taken as defining the number of solutions in the neighborhood of the origin.

The degree N of the eliminating polynomial is equal to the product of the degrees of the leading homogeneous polynomials of the r series $F_i(x_1, x_2, \dots, x_r; 0, \dots, 0)$ provided the algebraic resultant of these homogeneous polynomials is not zero.†

Relative to the case $r = 2$, we are now able to state the *Theorem*: If

$$(22) \quad F(x, y; z_1, \dots, z_s), \quad G(x, y; z_1, \dots, z_s)$$

are power series in the indicated variables, converging in a neighborhood of the origin and vanishing at that point, then the degree of the eliminating polynomial with respect to the dependent variables x, y is equal to the order of the resultant of the two series

$$(23) \quad \begin{aligned} f(x, y) &\equiv F(x, y; 0, \dots, 0) \\ g(x, y) &\equiv G(x, y; 0, \dots, 0). \end{aligned}$$

* Cf. Osgood, *Madison Colloquium Lectures* (1913), p. 194; and *Lehrbuch der Funktionentheorie II*, p. 107.

† Cf. Bliss, "A generalization of Weierstrass' Preparation Theorem for a power series in several variables," *Transactions of American Mathematical Society*, Vol. 13 (1912), p. 133.

The proof is essentially an adaptation of the method used by Professor Bliss * to obtain the eliminating polynomial, and we shall give only those details which have to do with the *degree* in question.

We first replace the series F and G by two formal series in the same variables, then delete certain terms so that the formal series corresponding to (23) will have the same orders respectively as $f(x, y)$ and $g(x, y)$, say m and n .

We now have two formal series which by proper specialization of coefficients will reduce to the series (22). To these two formal series we apply the transformation

$$\begin{aligned}x &= u_{11}\xi + u_{12}\eta \\y &= u_{21}\xi + u_{22}\eta,\end{aligned}$$

and to the two resulting series apply the Weierstrass Preparation Theorem, obtaining two polynomials

$$\begin{aligned}P &\equiv A_0\xi^m + A_1\xi^{m-1} + \cdots + A_m \\Q &\equiv B_0\xi^n + B_1\xi^{n-1} + \cdots + B_n.\end{aligned}$$

The coefficients A_0 and B_0 are independent of $\xi, \eta, z_1, \dots, z_s$, while A_i and $B_i (i > 0)$ are power series in z, z_1, \dots, z_s , vanishing with these variables.

The algebraic resultant of the polynomials P and Q is a power series which we may denote by $R(\eta; z_1, \dots, z_s)$. From the way in which this series was derived it follows that $R(\eta; 0, \dots, 0)$ is precisely the resultant series $r(\eta)$ of the two formal series from which (23) are obtained by specialization of the coefficients.

If we assign to all the literal coefficients in $R(\eta; z_1, \dots, z_s)$ the values they had in the special series (22), and to the resulting series apply the Weierstrass Preparation Theorem, we obtain a polynomial

$$E(\eta) \equiv \eta^N + e_1\eta^{N-1} + \cdots + e_N,$$

which is an eliminating polynomial for the two series (22). The degree N will be the order of the series $R(\eta; 0, \dots, 0)$ under the specialization of coefficients, that is will be the order of the resultant of the series (23).

UNIVERSITY OF SASKATCHEWAN.

* *Princeton Colloquium Lectures*, p. 70 et seq.

An Analysis of Logical Substitution.

By H. B. CURRY.

Contents.*

- I. Preliminary Discussion of the Nature of Mathematical Logic.
- II. Logical Substitution; its Relation to a Combinatory Problem.
- III. Solution of the Combinatory Problem.

I.

Mathematical Logic has been defined as an application of the formal methods of mathematics to the domain of Logic.† Logic, on the other hand, is the analysis and criticism of thought.‡ In accordance with these definitions, the essential purpose of mathematical logic is the construction of an abstract (or strictly formalized) theory, such that when its fundamental notions are properly interpreted, there ensues an analysis of those universal principles in accordance with which valid thinking goes on. The term *analysis* here means that a certain rather complicated body of knowledge is exhibited as deriveable from a much simpler body assumed at the beginning. Evidently the simpler this initial knowledge, and the more explicitly and carefully it is set forth, the more profound and satisfactory is the analysis concerned.

In the present paper I propose to take some preliminary steps toward a theory of logic in which the assumed initial knowledge is simpler than in any existing theory with which I am acquainted. Before this is done, however, it is necessary to consider somewhat in detail what is meant by the phrase "abstract theory," and what is the significance of such a theory for the analysis of thought. The object of this discussion is to see just how the assumed knowledge enters into the theory; for this purpose we shall need to be explicit, even at the risk of repeating what has already been better said by others.

Certain ideas concerning the nature of an abstract theory can be disposed

* The three parts of this paper are to a certain extent independent of one another. However, certain definitions needed in Part III are given in the last six paragraphs of Part II.

† Hilbert, D., and Ackermann, W., *Grundzüge der theoretischen Logik*, 1928, p. 1.

‡ Johnson, W. E., *Logic*, Part I, Cambridge (1921), p. xiii.

of at once. In the first place the naive notion that such a theory consists of a set of primitive ideas and propositions together with their consequences by the laws of pure logic, must be dismissed on the ground of its circularity. Again it is said that an abstract theory is one from which all meaning has been abstracted. This requires that the sense of the term meaning be explained. If we take the term meaning, as applied to objects, to signify the totality of properties (of those objects) which are directly apprehensible to our intuition, then every object presented to the mind has meaning, and a meaningless theory is a contradiction in terms. Even a symbol cannot be meaningless in this sense; for either it denotes some object, or else it is itself the object, and so has meaning. If we use the word meaning in some other sense, then it loses its significance as related to the assumed initial knowledge of our theory. Consequently the idea of a meaningless theory must be subjected to further scrutiny.

Let us use the word meaning, as applied to concepts, in the sense of the preceding paragraph. Then, relative to a given theory, we may distinguish two kinds of meanings, which we shall call natural and conventional meanings respectively. Natural meanings are those which are comprehensible a priori in terms of our previous knowledge; conventional meanings those based on relations to the theory itself. Natural meanings we may further subdivide into essential and accidental: essential meanings are those on which the deduction of the theory depends; accidental meanings those which are non-essential. The distinction between these three kinds of meaning is important in what follows.

The distinction between natural and conventional meanings has a counterpart in that between statements of fact and statements of convention. By a statement of fact I mean something of which truth or falsehood can significantly be predicated; by a statement of convention a declaration of intention, definition, or the like. The former corresponds to an act of judgment, the latter to one of volition. Common sense and grammar have long recognized both of these types; yet logicians seem to belittle the latter in that they define the proposition so as to exclude it.* Both these kinds of statement, however, are equally intelligible to a rational mind; in this sense it is false to say that one of them is less significant than the other. As examples of statements of convention we have of course the definitions of technical terms; but not all statements of convention are verbal—for instance the rules of chess, which, by a sufficient amount of circumlocution, may be stated

* See Johnson, W. E., *l. c.*, p. 1. Johnson's definition of the proposition is what I have given as the definition of a statement of fact.

without defining any new terms whatever. The postulates of any branch of mathematics are of this character.

Let us now return to the abstract theory. I suggest that such a theory is characterized by the following: 1) the explicit indication of all essential meanings; 2) the absence, or at least omission from consideration, of accidental meanings; 3) the circumstance that the statements with which the theory begins are conventional, and are, furthermore, sufficiently detailed so that all the acts necessary to the deduction are specified.

To be yet more precise, an abstract theory begins with a set of primitive notions, which, taken collectively, we shall call the *primitive frame*, as follows:

I. NON-FORMAL PRIMITIVE IDEAS.*

A set of ideas to each of which a certain amount of essential meaning is attached, although they need not coincide with any ideas previously entertained.† For example:

1. *Entities.* In order for an object to be considered in the theory at all, it must have some property; this fact we may express by saying it is an entity of one sort or another. These properties must then be among the primitive ideas of the theory; and they must have essential meaning in that they are predicates. In the simplified theory only one such notion is necessary; but in the more complicated ones there are several; e. g. in the *Principia Mathematica* there are individual, proposition, function, etc., the latter two of various orders and types.

2. *Modes of Combination.* I. e. processes by means of which entities may be combined to get new entities. These have essential meaning in that they are combinations. It must be specified by rules that the results of combination are entities. In the simple cases only one such notion is necessary, and that a dyadic one; in the more complicated cases the various processes of substitution are of this nature.

3. *Assertions.* An assertion is a kind of entity, which is of special importance because the object of deduction is to derive new assertions. The idea of assertion has essential meaning only in that it is a predicate applicable to certain entities. Ordinarily an assertion is interpreted as a statement to which belief attaches, but this meaning is accidental.

* The term *idea* is used here to denote an object, not a process of thought.

† I. e., their meaning may be partly conventional.

II. FORMAL PRIMITIVE IDEAS.

Ideas which have no essential meanings (except that they are concepts). They must of course be entities and their relations to other parts of the primitive frame will give them conventional meanings.

III. POSTULATES.

All propositions of the theory are statements that certain particular entities are assertions; the postulates are the propositions, if any, which are assumed at the beginning. They are purely conventional.

IV. RULES.

Statements of the processes by means of which new entities* or new propositions may be constructed. Such statements are of course conventional; moreover they are universal statements (involving the notion of "every" or its equivalent †). They thus differ from propositions not only in that they involve intuitive ideas from which the propositions are free, but also in that they form the methods of transition, rather than the stopping places in the theory. A typical example is the "rule of inference" which may be stated thus: whenever p and $p \supseteq q$ are assertions, then q shall also be an assertion.

In addition to the above notions there are yet to be considered those associated with the use of symbolism. Whether these are to be regarded as a part of the theory or as something superposed upon it, is a question which I prefer to leave to the reader to adopt such views as seem best to him. However he may decide, certainly language is necessary in order that the theory may be communicated. The use of this language may involve intuitive operations other than those we have mentioned; it is desirable that these, too, be specified by rules; because otherwise it is not certain that intuitive, knowledge, other than that expressly mentioned, does not creep into the theory. We shall call such rules *symbolic conventions*.

* Strictly speaking we should consider in the theory not only statements that an entity is an assertion, but also statements that such and such combinations are entities. But the latter are, in simple cases at least, of so trivial a nature that it is not necessary to give them special prominence.

† Otherwise the rule would make possible the addition of only a finite number of constituents, and these could just as well be added explicitly to the preceding categories of the primitive frame.

So much for the primitive frame. The abstract theory itself may now be defined as the doctrine built upon such a primitive frame by means of the following processes: 1) the derivation of new propositions, each of which is of the form that such and such an entity is an assertion, by means of the rules; 2) the addition of new ideas by definitions. The latter process may be regarded either as a symbolic matter, governed by symbolic conventions, or as the introduction of a new idea along with postulates and rules to the effect that it is identical with some already existing entity. It is worth emphasizing that since statements that entities are not assertions do not occur among the propositions, such a theory can never lead to a contradiction there.

The importance of such a theory for the analysis of thought lies in the definiteness with which the intuitive knowledge entering into it is set forth. Indeed, so far as the abstract theory itself is concerned, the only knowledge assumed is the appreciation of the essential meanings and conventional statements appearing in the primitive frame. When the theory is interpreted the additional knowledge that must be brought to bear consists of the following: that the concepts which we substitute for the primitive ideas have the necessary essential meanings, and that the conventional statements in the primitive frame correspond to facts. In both cases the required information is precisely specified.

On the philosophic nature of such a theory, its relations to the symbolism used in its expression, and to the various concrete theories obtained by interpreting it, it suffices to say that such questions are largely metaphysical, and therefore irrelevant to the present discussion. It is by no means self-evident that the best interests of science are served by adopting any one theory to the exclusion of all others; any more than it is desirable that two persons following the same argument should have the same mental imagery.*

The next point to which I wish to direct the reader's attention is the cardinal importance of the rules in any abstract theory related to logic. For the amount of initial knowledge which enters into the first three categories of the primitive frame is slight. In the rules, however, such knowledge is involved in every step of the construction; for we have to pass judgment as to whether the contemplated act is or is not according to Hoyle. These judgments, moreover, are the only ones which are required. The rules, therefore, form the port of entry of intelligence; and since nothing can be done without them,

* In writing the foregoing account I have naturally made use of any ideas I may have gleaned from reading the literature. The writings of Hilbert are fundamental in this connection. I hope that I have added clearness to certain points where the existing treatments are obscure.

they represent the atoms of thought, so to speak, into which the reasoning can be decomposed. It follows that in constructing such a theory it is not sufficient merely to reduce the postulates and primitive ideas to their lowest terms; it is even more important to so chose the rules that they involve, in their application, only the simplest actions of the human mind.

Now although the rule of inference, stated above, is simple enough, yet in all current mathematical logics there exist rules which are highly complex. The presence of these complex rules raises the question whether it is possible to formulate a theory which is—1) adequate for the whole of logic, 2) based on a finite number of primitive ideas, postulates, and rules, the last of the same order of complexity as the rule of inference. I believe that it is; indeed steps in that direction have already been taken.* As a preliminary to treating this general problem, I shall discuss in the rest of this paper a special one connected with it; viz., the analysis of the process of substitution. The latter process is one of those complicated rules which occur in practically every logical theory to-day.

The reader will observe that in the theory which results from the analysis the formulas are more complicated, and the deductions required to produce them more lengthy, than would be the case in the older theory. This is inevitable. Indeed if we are to dissect the reasoning into microscopic pieces it is but natural that more of them should be necessary to bring about a given result. Consequently we must adopt a point of view suggested by Hilbert. With each theory there is associated a metatheory in which we reason intuitively about the theory. In this metatheory we can derive more and more complicated rules by showing, in general terms, how any particular consequence of the derived rules can actually be deduced from the primitive ones. The aim of mathematical logic is, in fact, not to reduce mathematics to a formalism, à la *Principia*, in which all steps explicitly appear; but rather to analyze logic with a view to obtaining a greater command over its use, and a more profound understanding of its nature. In this paper we shall adopt this metatheoretic point of view.

II.

The process of substitution referred to above is the insertion of a constant entity for one or more of the variables in a propositional function. The complexity of this process is manifest. For not only is a function of n variables a distinct concept for every value of n , but the constant may be inserted in

* See the paper of Schönfinkel cited below.

any one of the n places, and each such insertion is a distinct act; furthermore, in connection with the universal and existential prefixes, we have a process which virtually amounts to the simultaneous substitution of an entity in two or more distinct places, as a procedure distinct from any of the foregoing. The process of substitution is therefore compound in that it is not one maneuver, but many. Moreover, when these acts of substitution are performed in succession, there are many equivalences between the different possibilities. To take the simplest example: suppose in a given function $\phi(x, y)$ we substitute a for x , and in the result, which is $\phi(a, y)$, we substitute b for y , we have the proposition $\phi(a, b)$; on the other hand, if we first substitute b for y and then a for x , we obtain the very same proposition. Yet these two processes are in no sense identical. The substitution process is therefore not only compound but complex, in the sense that it has structure. Thus there is a considerable amount of information presupposed by the process; and the rules involving it cannot have the maximum possible simplicity.

A notion closely related to substitution is that of transformation of functions. Suppose we regard a function as having inherent in its definition a certain order of its variables. Then permuting these variables in any way, or making two or more of them alike, will produce new functions related to the old; let us call them transforms of the original function, and the operations by which they are produced transformations. If we number the variables consecutively 1, 2, 3, . . . , then the transforms for a function of two variables will be—

$$\phi(1, 2), \phi(2, 1), \phi(1, 1).$$

For three variables there will be 13 transforms, for four variables 75, for five variables 541, etc. It is clear that the process of substituting a series of constants in an arbitrary manner (such that the total number of entities counting repetitions is n) into the original function is equivalent to the substitution of the same entities in a prescribed manner (viz., the first entity into the place of the first variable, the second into that of the second, etc.) into one of the transforms. The study of substitution is thus to a certain degree equivalent to the study of these transformations.

An important step toward the analysis of this situation was made by M. Schönfinkel.* Starting, apparently, from the fact that every logical formula is a combination of constants—the variables being only apparent—he shows that neither the notions of propositional function (of various orders)

* "Ueber die Bausteine der mathematischen Logik," *Mathematische Annalen*, Vol. 92 (1924), pp. 305-316.

nor that of substitution need be assumed as primitive; his formulation of logic is such that variables, real or apparent, do not appear explicitly. His primitive frame is essentially as follows: *

I. NON-FORMAL PRIMITIVE IDEAS.

1. *Entity*—not mentioned by Schönfinkel, but to be understood essentially as a single notion of the sort mentioned in the general description of an abstract theory above.

2. *Application*—a mode of combination, the only one in the theory. Two entities x and y combine to give a third entity called the application of x to y and denoted by (xy) . The *interpretation* of this is as follows: if x is a function, then (xy) is the result of substituting y for the first variable in x ; thus if f denote a function of one variable, (fx) denotes what is ordinarily written $f(x)$, if f is a function of two variables, $((fx)y)$ denotes what is ordinarily written $f(x, y)$, etc. Nothing is said concerning the interpretation of (xy) when x is not a function; if the reader is disturbed over this lack, he may invent one arbitrarily, *e. g.* (xy) may then be equal to x .

3. *Assertion*. To be understood as in the general description of an abstract theory. Not denoted by any particular symbol; but when a symbol of the form $((=)xy)$ (or $x=y$) where x and y may be quite complicated, stands out by itself like an equation in algebra, then the proposition that the corresponding entity is an assertion is to be understood.

II. FORMAL PRIMITIVE IDEAS.

Three, denoted by $(=)$, S and K . In the interpretation $(=)$ is to correspond with identity; S and K are operations in the sense defined by the rules.

Symbolic Conventions.

1. If x and y are any entities whatever, then instead of $((=)xy)$ we may write $(x=y)$.

2. If x_1, x_2, \dots, x_n are any entities, then instead of

$$((\dots((x_1 x_2)x_3)x_4)\dots)x_n)$$

we may write $(x_1 x_2 x_3 \dots x_n)$.

3. The outside parentheses may be left off in the case of a symbol standing by itself or on either side of the sign $=$.

* In this presentation I have changed Schönfinkel's formulation in some matters of detail.

III. POSTULATES.

None.

IV. RULES.

0. If x and y are entities, then (xy) shall be an entity.

1. $(=)$ shall have the properties of identity. These properties may be specified by a few simple rules; but in this treatment we shall not go into that detail. We shall treat $(=)$ as if it were precisely the intuitive relation of equality.

2. If x and y are any entities, then

$$Kxy = x$$

3. If x, y, z are entities, then

$$Sxyz = xz(yz)$$

4. If X and Y are combinations of S and K , and if there exists an integer n such that by application of the preceding rules we can formally reduce the expressions $Xx_1 x_2 \cdots x_n$ and $Yx_1 x_2 \cdots x_n$ to combinations of $x_1 x_2 \cdots x_n$ which have the same structure, then $X = Y$.

If the above primitive frame were a part of a general theory of logic, the term entity would include not only the various combinations of S and K , but all the notions of logic as well. In the sequel we shall accordingly speak of the application of combinations S and K to various logical notions, and of the resulting notions to each other, just as if these notions had been adjoined to the above frame.

The *raison d'être* of the theory based on this frame is the following fact: Let x_1, x_2, \cdots, x_n be any n entities, and X any combination of them constructed by means of application. Then there exists a unique Y , which is a combination of S and K and independent of x_1, x_2, \cdots, x_n such that

$$X = Yx_1 x_2 \cdots x_n.$$

When we recall the interpretation to be given to application we have the following result: given any logical formula built up from functions f_1, f_2, \cdots, f_m and variables x_1, x_2, \cdots, x_n by substitution and rearrangement in any manner; then the formula is expressible in the form

$$Fx_1 x_2 \cdots x_m$$

where

$$F = Yf_1 f_2 \cdots f_m.$$

Now as already remarked, in the formulas expressing propositions of logic, the variables are only apparent; which means that they are only a device by means of which rather complicated relations among the logical constants may be expressed; these relations, as the preceding argument shows, may also be expressed by means of the operators Y , so that when the Schönfinkel theory is used it is not necessary that variables should appear at all.

The theory of transformation of functions is included in the above as a special case; viz., when there is a single function f and F is a transform.

The theory is, however, open to objection from our point of view because of the complexity of Rule 4. This rule is not mentioned by Schönfinkel; but it is necessary in order that the Y mentioned be unique. In fact, the combinations SK and $K(SKK)$ determine the same X ; yet it is evidently not possible to establish their identity by means of the first four rules.

This situation suggests a problem; viz., to find a set of postulates which, when adjoined to the Schönfinkel frame, enable us to dispense with Rule 4. In what follows we shall obtain the solution of a special case under this problem; specifically, we shall find a set of postulates such that, within a certain subclass of combinations of S and K , all the Y 's which correspond to the same X may be proved equal by means of these postulates and Rules 0-3. The subclass is one which has particular reference to logical substitution.

To begin with, we make the following definitions (the first three were made by Schönfinkel): *

$$\begin{aligned} I &= SKK \\ B &= S(KS)K \\ C_1 &= S(BBS)(KK) \\ C_2 &= BC_1; C_3 = BC_2; \cdots \text{etc.} \\ W &= SS(SK) \end{aligned}$$

then the reader may verify that whenever $x_0, x_1, x_2 \cdots$ are entities

$$\begin{aligned} Ix_0 &= x_0 \\ Bx_0 x_1 x_2 &= x_0(x_1 x_2) \\ C_1 x_0 x_1 x_2 &= x_0 x_2 x_1 \\ C_2 x_0 x_1 x_2 x_3 &= x_0 x_1 x_3 x_2 \\ &\cdots \cdots \cdots \\ Wx_0 x_1 &= x_0 x_1 x_1 \end{aligned}$$

* We use B and C respectively in place of Schönfinkel's Z and T . Nothing corresponding to W or C_2, C_3, \cdots is defined by him.

Also we define multiplication, thus

$$X \cdot Y = BXY$$

This multiplication is associative, and furthermore with respect to it, I is an identical element. These properties follow from Rule 4; they may also be proved from the postulates:

$$C_1(BB(BBB))B = BBBBB$$

$$BI = I$$

$$C_1 BI = I.$$

The subclass in question shall now consist of all the combinations formed by multiplication from the C 's, W , and I .

Before discussing the significance of this subclass, we turn aside to make some conventions in regard to transformations. Let us redefine a transformation as an operation converting the sequence $1, 2, 3, \dots$ into the sequence a_1, a_2, a_3, \dots . The latter sequence will, in the cases we consider, have the property that there exists an index m such that: 1) for $i \leq m$ the a_i are a permutation, with repetitions allowed, but not omissions, of the integers from 1 to $m - p$, $p \geq 0$, inclusive, 2) for $i > m$, $a_i = i - p$. Let m be the least integer having this property; then we shall call m the order of the transformation. We shall denote a transformation by writing in brackets the sequence into which it transforms the sequence $1, 2, 3, \dots$; furthermore, there is no ambiguity if we indicate only the first n terms of the former sequence, if $m \leq n$.

We shall further agree that a transformation of order m may operate on a function of any number of variables $\geq m$; and that the effect of the transformation on the function $\phi(1, 2, \dots, n)$, where $n \geq m$, shall be the transform $\phi(a_1, a_2, \dots, a_n)$. We shall regard as undefined the effect of a transformation on a function of multiplicity less than the order of the transformation. Then there is one and only one transformation which carries a given function into a given transform.

Multiplication of transformations we now define as follows:

$$(1) \quad [a_1, a_2, a_3, \dots] \cdot [b_1, b_2, b_3, \dots] = [a_{b_1}, a_{b_2}, a_{b_3}, \dots]$$

(The product transformation has a finite order, provided that the factors do.)

Suppose, now, that we have a combination, X , of S and K , having the property that there exists a transformation $\alpha = [a_1, a_2, \dots, a_m]$ such that for arbitrary $x_0, x_1, x_2, \dots, x_{m-p}$

$$X x_0 x_1 x_2 \dots x_{m-p} = x_0 x_{a_1} x_{a_2} \dots x_{a_m}$$

where m and p are defined as above. Then we shall say that X corresponds to α . It follows from Rules 0-3 that if X and Y are entities which correspond to transformations α and β , respectively, then $X \cdot Y$ corresponds to $\alpha \beta$.

I now assert that every entity of our subclass corresponds to a transformation; and conversely that every X corresponding to a transformation, is an entity of the subclass. For since the generating entities, the C 's, W , and I , correspond to transformations, the first part of the statement follows from the closing sentence of the last paragraph. The last part follows similarly from the fact that every transformation can be obtained by multiplication from permutations of adjacent integers and the transformation $[1, 1]$. Of course the one-to-oneness of the correspondence depends essentially on Rule 4.

The problem now is to find a set of postulates for the C 's and W , from which we may conclude, without the use of Rule 4, that the correspondence between our subclass and the set of all transformations is a simple isomorphism. To this we now turn.

III.

Theorem

1) Let \mathfrak{A} be a system of operators, in which there exists an associative multiplication and an identical element, I .

2) Let this system be generated by operators W, C_1, C_2, \dots . Subject to the postulates

- I. $C_i \cdot C_i = I$ $i = 1, 2, 3, \dots$
- II. $C_i \cdot C_{i+1} \cdot C_i = C_{i+1} \cdot C_i \cdot C_{i+1}$ $i = 1, 2, 3, \dots$
- III. $C_i \cdot C_j = C_j \cdot C_i$ $i = 1, 2, 3, \dots \quad j > i + 1$
- IV. $C_i \cdot W = W \cdot C_{i+1}$ $i = 2, 3, 4, \dots$
- V. $W \cdot C_1 = W$
- VI. $W \cdot W \cdot C_2 = W \cdot W^*$
- VII. $W \cdot D_j \cdot W = D_{j-1} \cdot W \cdot D_3 \cdot W \cdot C_2 \cdot C_3 \cdot C_1 \cdot C_2^* \quad j = 3, 4, 5, \dots$

where

$$D_1 = I$$

$$D_{j+1} = C_j \cdot D_j = C_j \cdot C_{j-1} \cdot C_{j-2} \cdot \dots \cdot C_1$$

3) Let a correspondence be set up between the system \mathfrak{A} and the system of transformations of finite order as follows:

* VI and VII are equivalent respectively to (6) and (7) (see below). The latter may if desired replace VI and VII.

$$\begin{aligned}
 (2) \quad & W \sim [1, 1] \\
 & C_1 \sim [2, 1] \\
 & C_2 \sim [1, 3, 2] \\
 & C_i \sim [1, 2, 3, \dots, i-1, i+1, i]
 \end{aligned}$$

and if $A \sim \alpha$, $B \sim \beta^*$ then $A \cdot B \sim \alpha \cdot \beta$ where $\alpha \cdot \beta$ is defined by (1).

Then, the correspondence so defined is a one-to-one isomorphism.

Proof.

There are four things to prove:

- 1) That to every product expression in \mathfrak{M} there corresponds a unique transformation.
- 2) That this correspondence is an isomorphism.
- 3) That if two transformations correspond to two expressions in \mathfrak{M} which are equal by Virtue of I-VII (inclusive), then these transformations are equal.
- 4) That conversely there corresponds an operator of \mathfrak{M} to each transformation, and any two operators corresponding to the same transformation may be proved equal by I-VII.

Of these four things the first three follow immediately: the first two by definition; the third because propositions analogous to I-VII are true for transformations.†

It remains, therefore, only to prove the fourth.

By a theorem due to E. H. Moore‡ postulates I-III insure that the subset of \mathfrak{M} which is obtained from the first $n-1$ C 's only is isomorphic

* Throughout this discussion we use Roman capitals to denote operators of \mathfrak{M} , Greek l. c. letters to denote transformations.

† The only question here is about VII. It follows, however, by definition, that

$$D_i \sim [i, 1, 2, 3, \dots, i-1]$$

and therefore the proposition analogous to VII is the identity

$$\begin{aligned}
 & [1, 1] \cdot [j, 1, 2, \dots, j-1] \cdot [1, 1] \\
 & \quad = [j-1, 1, 2, \dots, j-2] \cdot [1, 1] \cdot [3, 1, 2] \cdot [1, 1] \cdot [3, 4, 1, 2].
 \end{aligned}$$

Both sides here are equal to $[j-1, j-1, 1, 1, 2, 3, \dots, j-2]$.

‡ *Proceedings of the London Mathematical Society*, Vol. 28 (1897), p. 357-366. Moore's proof involves some rather complicated theorems in group theory. It is, however, possible to prove the theorem by directly establishing the correspondence.

Proof.

First we show that any expression of the form $A \cdot W$ can be reduced to the form $W_k \cdot B$, where A and B involve the C 's only. If A is the identical element we are through. If not, consider the C nearest the W ; if this is any other than C_1 it can be passed across the W by IV, otherwise we have an expression of the form $A' \cdot C_1 \cdot W$ where A' involves one less C than A .

Now suppose we have reduced $A \cdot W$ to the form $A' \cdot D_j \cdot W \cdot B'$ where A' and B' involve the C 's only and A' is not I . Let C_i be that C in A' which is nearest D_j . Four cases can arise: 1) if $i < j-1$, $C_i \cdot D_j = D_j \cdot C_{i+1}$, and since $i+1 > 1$, C_{i+1} can be passed across the W by IV; 2) if $i = j-1$, $C_i \cdot D_j = D_{j-1}$; 3) if $i = j$, $C_i \cdot D_j = D_{j+1}$; 4) if $i > j$, $C_i \cdot D_j = D_j \cdot C_i$, and since $i > j \geq 1$, C_i can be passed across the W by IV. In all four cases we have reduced the expression to another one of the same form where the new A' involves one less C than the old.

Continuing in this way we must eventually reach a stage where $A' = I$. Then we have

$$\begin{aligned} A \cdot W &= D_k \cdot W \cdot B' \\ &= (D_k \cdot W \cdot C_2 \cdot C_3 \cdots C_k \cdot C_1 \cdot C_2 \cdots C_{k-1}) \\ &\quad \cdot C_{k-1} \cdots C_2 \cdot C_1 \cdot C_k \cdots C_3 \cdot C_2 \cdot B' \text{ by I} \\ &= W_k \cdot C_{k-1} \cdots C_2 \cdot C_1 \cdot C_k \cdots C_3 \cdot C_2 \cdot B' \text{ by def.} \end{aligned}$$

which is of the form $W_k \cdot B$.

The proof of the lemma now follows. Let the given expression be of the form

$$(4) \quad A_q \cdot W \cdot A_{q-1} \cdot W \cdots A_2 \cdot W \cdot A_1 \cdot W \cdot A_0$$

where $A_0, A_1, A_2 \cdots A_q$ involve the C 's only. By what we have just proved,

$$\begin{aligned} A_q \cdot W &= W_{k_q} \cdot B_q \\ B_q \cdot A_{q-1} \cdot W &= W_{k_{q-1}} \cdot B_{q-1} \\ B_{q-1} \cdot A_{q-2} \cdot W &= W_{k_{q-2}} \cdot B_{q-2} \\ &\vdots \\ B_2 \cdot A_1 \cdot W &= W_{k_1} \cdot B_1 \\ B_1 \cdot A_0 &= B. \end{aligned}$$

Taking all these into consideration, we can reduce the form (4) to the form (3).

LEMMA 2. *Every expression in \mathfrak{A} can be reduced further to a form (3), in which*

$$k_q \geq k_{q-1} \geq \cdots \geq k_2 \geq k_1$$

Proof. It is sufficient to prove that

$$(5) \quad W_j \cdot W_{k+1} = W_k \cdot W_j \quad \text{for } k \geq j.$$

The proof of (5) is as follows.

First,

$$(6) \quad \begin{array}{ll} W_1 \cdot W_2 = W_1 \cdot W_1. \\ \text{For } W_1 \cdot W_2 = W_1 \cdot C_1 \cdot W_1 \cdot C_2 \cdot C_1 & \text{by def.} \\ & = W_1 \cdot W_1 \quad \text{by V and VI.} \end{array}$$

Second,

$$(7) \quad \begin{array}{ll} W_1 \cdot W_{k+1} = W_k \cdot W_1 & \text{for } k > 1. \\ \text{For } W_1 \cdot W_{k+1} = W_1 \cdot D_{k+1} W_1 \cdot [3, 4, \dots k+2, 1, 2] & \text{by def.} \\ & = D_k \cdot W_1 \cdot [3, 1, 2] \cdot W \cdot [3, 4, 1, 2] \\ & \quad \cdot [3, 4, \dots k+2, 1, 2] \quad \text{by VII.} \end{array}$$

$$\text{But since } [3, 4, 1, 2] \cdot [3, 4, \dots k+2, 1, 2] \\ = [1, 2, 5, 6, \dots k+2, 3, 4]$$

$$\text{and } W_1 \cdot [1, 2, 5, 6, \dots k+2, 3, 4] \\ = [1, 4, 5, \dots k+1, 2, 3] \cdot W_1 \quad \text{by IV.} \\ [3, 1, 2] \cdot [1, 4, 5, \dots k+1, 2, 3] \\ = [3, 4, \dots k+1, 1, 2]$$

we have

$$W_1 \cdot W_{k+1} = D_k \cdot W_1 \cdot [3, 4, \dots k+1, 1, 2] \cdot W \\ = W_k \cdot W_1 \quad \text{q. e. d.}$$

Third,

$$(8) \quad C_k \cdot W_j = W_j \cdot C_k \quad \text{for } k < j-1.$$

For by definition

$$C_k \cdot W_j = C_k \cdot D_j \cdot W \cdot [3, 4, \dots j+1, 1, 2].$$

But we have

$$\begin{aligned} C_k \cdot D_j &= D_j \cdot C_{k+1} \\ C_{k+1} \cdot W &= W \cdot C_{k+2} \\ C_{k+2} \cdot [3, 4, \dots j+1, 1, 2] &= [3, 4, \dots j+1, 1, 2] \cdot C_k \end{aligned}$$

(the first and third relations follow from the known properties of permutations, the second from IV.) Combining these three we have (8).

Fourth,

$$(9) \quad W_j \cdot C_j = W_j.$$

For $j = 1$, this is true by V.

Suppose, now, (9) is true for $j-1$ (i. e. suppose $W_{j-1} \cdot C_{j-1} = W_{j-1}$) then

$$\begin{aligned} W_j \cdot C_j &= C_{j-1} \cdot W_{j-1} \cdot C_j \cdot C_{j-1} \cdot C_j && \text{by def.} \\ &= C_{j-1} \cdot W_{j-1} \cdot C_{j-1} \cdot C_j \cdot C_{j-1} && \text{by II.} \\ &= C_{j-1} \cdot W_{j-1} \cdot C_j \cdot C_{j-1} && \text{by hyp.} \\ &= W_j && \text{by def.} \end{aligned}$$

Fifth,

$$(10) \quad C_{j+h} \cdot W_j = W_j \cdot C_{j+h+1} \quad \text{for } h > 0$$

For $j=1$, this is true by IV. Suppose it true for $j-1$, then

$$\begin{aligned} C_{j+h} \cdot W_j &= C_{j+h} \cdot C_{j-1} \cdot W_{j-1} \cdot C_j \cdot C_{j-1} && \text{by def.} \\ &= C_{j-1} \cdot C_{j+h} \cdot W_{j-1} \cdot C_j \cdot C_{j-1} && \text{by III} \\ &= C_{j-1} \cdot W_{j-1} \cdot C_{j+h+1} \cdot C_j \cdot C_{j-1} && \text{by hyp.} \\ &= C_{j-1} \cdot W_{j-1} \cdot C_j \cdot C_{j-1} \cdot C_{j+h+1} && \text{by III} \\ &= W_j \cdot C_{j+h+1} && \text{q. e. d.} \end{aligned}$$

Sixth,

$$W_j \cdot W_{j+1} = W_j \cdot W_j$$

For $j=1$, this is true by (6). Suppose it true for $j-1$, then

$$\begin{aligned} W_j \cdot W_j &= C_{j-1} \cdot W_{j-1} \cdot C_j \cdot W_{j-1} \cdot C_j \cdot C_{j-1} && \text{by def. and I.} \\ &= C_{j-1} \cdot W_{j-1} \cdot W_{j-1} \cdot C_{j+1} \cdot C_j \cdot C_{j-1} && \text{by (10)} \\ &= C_{j-1} \cdot W_{j-1} \cdot W_j \cdot C_{j+1} \cdot C_j \cdot C_{j-1} && \text{by hyp.} \\ &= C_{j-1} \cdot W_{j-1} \cdot C_j \cdot C_{j-1} \cdot C_{j-1} \cdot C_j \cdot W_j \cdot C_{j+1} \cdot C_j \cdot C_{j-1} && \\ & && \text{by I} \\ &= W_j \cdot C_{j-1} \cdot W_{j+1} \cdot C_{j-1} && \text{by def.} \\ &= W_j \cdot W_{j+1} && \text{q. e. d.} \end{aligned}$$

Seventh,

$$W_j \cdot W_{k+1} = W_k \cdot W_j \quad \text{for } k > j$$

For $j=1$, this is true by (7). Suppose it true for $j-1$, then,

$$\begin{aligned} W_k \cdot W_j &= W_k \cdot C_{j-1} \cdot W_{j-1} \cdot C_j \cdot C_{j-1} && \text{by def.} \\ &= C_{j-1} \cdot W_k \cdot W_{j-1} \cdot C_j \cdot C_{j-1} && \text{by (8), since } j-1 < k-1 \\ &= C_{j-1} \cdot W_{j-1} \cdot W_{k+1} \cdot C_j \cdot C_{j-1} && \text{by hyp.} \\ &= C_{j-1} \cdot W_{j-1} \cdot C_j \cdot C_{j-1} \cdot W_{k+1} && \text{by (8)} \\ &= W_j \cdot W_{k+1} && \text{q. e. d.} \end{aligned}$$

Definition: $W_k^r = W_k \cdot W_k \cdot \dots \cdot W_k$ r times.

COROLLARY. Every expression in \mathfrak{A} can be reduced to the form

$$(11) \quad W_{k_q}^{r_q} \cdot W_{k_{q-1}}^{r_{q-1}} \cdot \dots \cdot W_{k_2}^{r_2} \cdot W_{k_1}^{r_1} \cdot B$$

where $k_q > k_{q-1} > \dots > k_2 > k_1, \quad r_i > 0$

Proof. Simply collect the W 's with equal indices in (3).

LEMMA 3. *If two expressions of the form (11) correspond to the same transformation, then they both have the same constants $q, k_1, k_2, \dots, k_q, r_1, r_2, \dots, r_q$.*

Proof.

$$\text{Let} \quad \Delta_q = W_{k_q}^{r_q} \cdot W_{k_{q-1}}^{r_{q-1}} \cdot \dots \cdot W_{k_2}^{r_2} \cdot W_{k_1}^{r_1}$$

$$\text{also let} \quad \Delta_q \sim \alpha = [a_1, a_2, a_3 \cdot \dots \cdot]$$

Then for the explicit determination of the a_x we have

$$\begin{array}{ll} \text{If} & 0 < x < k_1 & \text{then} & a_x = x \\ & k_1 \leq x \leq k_1 + r_1 & & a_x = k_1 \\ & k_1 + r_1 < x < k_2 + r_1 & & a_x = x - r_1 \\ & & \text{etc.} & \end{array}$$

In general, if we define

$$\begin{aligned} s_i &= r_1 + r_2 + \dots + r_i \\ r_0 &= 0, \quad s_0 = 0, \quad k_0 = 0 \end{aligned}$$

then, for $i = 1, 2, \dots, q$.

$$\begin{array}{ll} \text{If} & k_{i-1} + s_{i-1} < x < k_i + s_{i-1} & \text{then} & a_x = x - s_{i-1} \\ & k_i + s_{i-1} \leq x \leq k_i + s_i & & a_x = k_i \\ & k_q + s_q < x & & a_x = x - s_q. \end{array}$$

These formulas follow directly from (1) and (2); their proofs are left to the reader.

The transformation has thus the following character: 1) the integers in the symbol $[a_1, a_2, a_3 \cdot \dots]$ are arranged in their natural order with certain repetitions, 2) the only integers which appear more than once are $k_1, k_2 \cdot \dots, k_q$, and these appear respectively $r_1 + 1, r_2 + 1, \dots, r_q + 1$, times.

Now if $\beta = [b_1, b_2 \cdot \dots]$ corresponds to B , then the transformation corresponding to $\Delta_q \cdot B$ is $\alpha \cdot \beta = [a_{b_1}, a_{b_2}, \dots]$. By the restrictions on B the sequence $[a_{b_1}, a_{b_2}, \dots]$ is merely a rearrangement of the sequence $[a_1, a_2 \cdot \dots]$. Hence the second of the above properties applies just as much to $\alpha \cdot \beta$ as to α . But then the constants $q, k_1 \cdot \dots, k_q, r_1 \cdot \dots, r_q$, are uniquely determined by $\alpha \cdot \beta$, q. e. d.

LEMMA 4. *To each transformation of finite order there corresponds at least one expression of form (11).*

Proof.

Let the given transformation be $\gamma = [c_1, c_2 \dots]$.

Let Δ_q be determined from γ as indicated in the preceding lemma, and suppose $\Delta_q \sim [a_1, a_2 \dots] = \alpha$.

Then we can construct a permutation $\beta = [b_1, b_2 \dots]$ such that $\alpha \cdot \beta = \gamma$, as follows. If c_i is distinct from any of the k 's there is one and only one j such that $a_j = c_i$; in that case we must let $b_i = j$. On the other hand let the C 's which are equal to k_j be $c_{i_1}, c_{i_2} \dots c_{i_\rho}$ (where $\rho = r_j + 1$); then set $b_{i_1} = k_j + s_{j-1}$, $b_{i_2} = b_{i_1} + 1$, $b_{i_3} = b_{i_2} + 1$, etc. For definiteness we may suppose that $i_1 < i_2 < \dots < i_\rho$. Then $\alpha \cdot \beta = \gamma$; for the i 'th integer in the symbol for $\alpha \cdot \beta$ is a_{b_i} , and b_i has been so chosen that in all cases $a_{b_i} = c_i$.

To the β so constructed there corresponds a unique B in \mathfrak{A} , by Moore's result. For this B , $\Delta_q \cdot B \sim \gamma$, q. e. d.

LEMMA 5. *The operator in \mathfrak{A} corresponding to a given transformation is unique.*

Proof.

Let A_1 and A_2 be two operators in \mathfrak{A} which correspond to a given transformation γ . By Lemmas 2 and 3 there is a uniquely determined Δ_q such that

$$A_1 = \Delta_q \cdot B_1, \quad A_2 = \Delta_q \cdot B_2$$

If $B_1 \sim \beta_1$, $B_2 \sim \beta_2$, β_1 and β_2 must be subject to all the restrictions to which β was subject in Lemma 4, except that it is not necessary to suppose $i_1 < i_2 < \dots < i_\rho$. The only way in which β_1 and β_2 can differ is in the arrangement of the $b_{i_1}, b_{i_2} \dots$ corresponding to each of the k_j . Hence

$$\beta_2 = \beta_3 \cdot \beta_1$$

where β_3 is a product of permutations, each of which permutes among themselves the integers composing one of the sets $k_j + s_{j-1}$, $k_j + s_{j-1} + 1$, \dots , $k_j + s_j$.

Let us now agree to denote by E_i^j , $i < j$ a combination of $C_i, C_{i+1} \dots C_{j-1}$

(corresponding to a permutation of $i, i+1, \dots, j$). Then, in view of the isomorphism already established by Moore, we have the following result; there exists a B_3 such that

$$B_2 = B_3 \cdot B_1$$

$$B_3 = E_{k_1}^{k_1+r_1} \cdot E_{k_2+s_1}^{k_2+s_2} \cdot E_{k_3+s_2}^{k_3+s_3} \cdot \dots \cdot E_{k_q+s_{q-1}}^{k_q+s_q}$$

To prove the lemma it is sufficient to show that $\Delta_q \cdot B_3 = \Delta_q$. This, in turn, follows from the above form for B_3 , if we demonstrate that

$$(12) \quad \Delta_h \cdot E_{k_h+s_{h-1}}^{k_h+s_h} = \Delta_k \quad (h = 1, 2, \dots, q).$$

We turn to this last question forthwith.

In the first place, if

$$i > k+1, W_k \cdot E_i^j = E_{i-1}^{j-1} \cdot W_k.$$

This follows from (10), under Lemma 2.

Next, if $i > k+r$

$$(13) \quad W_k^r \cdot E_i^j = E_{i-r}^{j-r} \cdot W_k^r.$$

This is derived from the preceding by induction on r .

Third, if $i < h$

$$(14) \quad \Delta_i \cdot E_{k_h+s_{h-1}}^{k_h+s_h} = E_{k_h+s_{h-1}-s_i}^{k_h+s_h-s_i} \cdot \Delta_i.$$

For $i=1$, this follows from the preceding, for the conditions $k_h + s_{h-1} > k_1 + r_1$ are satisfied since, if $h > 1$, $k_h > k_1$, $s_{h-1} \geq s_1 = r_1$. For $i > 1$ we prove (14) by induction. Suppose it true for $i-1$, then,

$$\begin{aligned} \Delta_i \cdot E_{k_h+s_{h-1}}^{k_h+s_h} &= W_{k_i}^{r_i} \cdot \Delta_{i-1} \cdot E_{k_h+s_{h-1}}^{k_h+s_h} \\ &= W_{k_i}^{r_i} \cdot E_{k_h+s_{h-1}-s_{i-1}}^{k_h+s_h-s_{i-1}} \cdot \Delta_{i-1}. \end{aligned}$$

In order, now, to apply (13), we need to know simply that

$$k_h + s_{h-1} - s_{i-1} > k_i + r_i$$

and this is fulfilled since for $h > i$,

$$s_{h-1} - s_{i-1} = r_i + r_{i+1} + \dots + r_{h-1} \geq r_i.$$

Hence

$$W_{k_i}^{r_i} \cdot E_{k_h + s_{h-1} - s_{i-1}}^{k_h + s_h - s_{i-1}} \cdot \Delta_{i-1} = E_{k_h + s_{h-1} - s_i}^{k_h + s_h - s_i} \cdot W_{k_i}^{r_i} \cdot \Delta_{i-1}$$

so that (14) is proved.

The following is the special case of (14) where $i = h - 1$

$$\Delta_{h-1} \cdot E_{k_h + s_{h-1}}^{k_h + s_h} = E_{k_h}^{k_h + r_h} \cdot \Delta_{h-1}.$$

In order to complete the proof of (12), it remains (since $\Delta_h = W_{k_h}^{r_h} \cdot \Delta_{h-1}$) simply to show

$$(15) \quad W_{k_h}^{r_h} \cdot E_{k_h}^{k_h + r_h} = W_{k_h}^{r_h}$$

In view of the composition of $E_{k_h}^{k_h + r_h}$, (15) follows from

$$W_k^r \cdot C_{k+j} = W_k^r \quad \text{for } 0 \leq j < r.$$

For $j = 0$, this is true by (9), under Lemma 2.

For $j = 1$

$$W_k \cdot W_k \cdot C_{k+1} = W_k \cdot W_{k+1} \cdot C_{k+1} \quad \text{by (5)}$$

$$= W_k \cdot W_{k+1} \quad \text{by (9)}$$

$$= W_k \cdot W_k \quad \text{by (5)}$$

$$\therefore W_k^r \cdot C_{k+1} = W_k^r \quad \text{for } r \geq 2.$$

For $j > 1$, let $r = s + j - 1$, then $s \geq 2$ and

$$\begin{aligned} W_k^r \cdot C_{k+j} &= W_k^s \cdot W_k^{j-1} \cdot C_{k+j} \\ &= W_k^s \cdot C_{k+1} \cdot W_k^{j-1} && \text{by (10)} \\ &= W_k^s \cdot W_k^{j-1} && \text{by the previous case} \\ &= W_k^r. \end{aligned}$$

These three cases together establish (15), which completes the proof of both the lemma and the theorem.

We have now achieved one of our objectives—viz. the elimination of Rule 4 so far as the selected subclass is concerned. For Rule 4 merely enables us to conclude that two entities of the subclass, whose application to a series $x_0 x_1 x_2 \dots$ yields the same transformation, are equal. These entities correspond to the same transformation in the sense of our theorem, and therefore by that theorem are equal. Moreover the proof of the theorem

is such that in any particular case a proof of the equality can be constructed in terms of I-VII, together with the rules for identity and no more. This proof will in general be quite long, but it may nevertheless be found.*

Finally we have an analysis of substitution in so far as that process is equivalent to transformation.

There remain open two questions: 1) What is the relation between the transforms of a function of $n - 1$ variables, obtained by substituting a constant in a function of n variables, and the transforms of the original; 2) How can our infinite set of postulates be reduced to a finite set. Both these questions can be answered by introducing the entity B defined in Part II. But when we do that, it is expedient to consider a more extensive subclass. This takes us out of the domain of simple substitution. The topic is left for a later paper.

* This statement is subject to reservation only in that the proof of Moore's theorem is indirect, but a direct proof may easily be found.

On Extending a Correspondence in the Sense of Antoine.*

BY HARRY MERRILL GEHMAN.

Let M and M' denote point sets which are subsets of the point sets S and S' respectively and let T be a continuous $(1-1)$ correspondence \dagger such that $T(M) = M'$; then if there exists a continuous $(1-1)$ correspondence U , such that $U(S) = S'$ and $U(M) = M'$, we shall say that the correspondence T can be *extended in the sense of Antoine* \ddagger to a correspondence between S and S' , or for brevity, T can be A -extended to a correspondence between S and S' .

Note that in order to be able to A -extend the correspondence T , it is not necessary that for each point P of M , $U(P)$ be the same as $T(P)$. That is, if the correspondence T can be *extended* in the sense in which we have used the word in a previous paper, \S it can also be A -extended, but not conversely.

If M and M' are plane continuous curves and S and S' are planes, it has been proved in the paper just cited that the correspondence T can be extended (and hence A -extended) to a correspondence between the planes S and S' provided that sides are preserved under T . If sides are not preserved, T cannot be extended, but can nevertheless be A -extended provided that there exists another continuous $(1-1)$ correspondence V , such that $V(M) = M'$ and such that sides are preserved under V . If the continuous curve M is sufficiently simple in its constitution, this correspondence V always exists and *every* correspondence T can be A -extended to a correspondence between the planes.

* Presented to the American Mathematical Society at Nashville, December 28th, 1927.

\dagger A correspondence T between M and $T(M)$ is said to be *continuous*, if in case the point P of M is a limit point of N , a subset of M , then $T(P)$ is a limit point of $T(N)$. Since all the correspondences considered in this paper are continuous and $(1-1)$, we shall frequently use "correspondence" in place of "continuous $(1-1)$ correspondence."

\ddagger L. Antoine, "Sur l'homéomorphie de deux figures et de leurs voisinages," *Journal de Mathématiques*, Ser. 8, Vol. 4 (1921), p. 221.

\S H. M. Gehman, "On Extending a Continuous $(1-1)$ Correspondence of Two Plane Continuous Curves to a Correspondence of Their Planes," *Transactions of the American Mathematical Society*, Vol. 28 (1926), pp. 252-265.

The first object of this paper is to determine for what classes of plane continuous curves (classified according to the number of end points,* branch points † and simple closed curves, and a certain interior property to be defined next) it is true that if a correspondence T exists between two curves of the same class, a side preserving correspondence V also exists between them, and hence we can A -extend the correspondence T to a correspondence between the planes.

A necessary, but not sufficient, condition that a correspondence preserve sides is that it preserve interiors of simple closed curves. Hence in order that V exist, it is necessary to impose some interior condition on the two continuous curves. The condition used here (that they be in the same interior class) is an extremely light one, as is shown by Example 6 below. This example shows that the condition that two continuous curves be in the same interior class is not sufficient for the correspondence between them to be A -extended. The condition is necessary, of course.

DEFINITION. We shall say that the two plane continuous curves M and M' are in the same interior class with respect to the planes S and S' in which they lie, if there exists (a) a continuous $(1-1)$ correspondence T such that $T(M) = M'$, and (b) a $(1-1)$ correspondence between the set of all simple closed curves in M and the set of all simple closed curves in M' , which is such that if J is a simple closed curve in M and J' is the corresponding simple closed curve in M' , and if N is the set of all points of M which are interior to J and N' is the set of all points of M' which are interior to J' , then there exists a continuous $(1-1)$ correspondence W , such that $W(N) = N'$.

THEOREM 1. If M is a continuous curve lying in a plane S , and T is a continuous $(1-1)$ correspondence such that $T(M) = M'$, where M' lies in a plane S' , then T can be A -extended to a correspondence between the planes S and S' , provided that (1) M and M' are in the same interior class with respect to S and S' , and (2) c , b , and e satisfy one of the following condi-

* For a number of equivalent definitions see: H. M. Gehman, "Concerning End Points of Continuous Curves and Other Continua," *Transactions of the American Mathematical Society*, Vol. 30 (1928), p. 63. Probably the simplest definition is one due to G. T. Whyburn: A point P of a continuous curve M is said to be an *end point* of M in case P is not an interior point of any arc in M .

† A point P of a continuous curve M is said to be a *branch point* of M in case M contains three arcs having P in common, but no two of them having any other point in common.

tions, where c is the number of simple closed curves in M , b is the number of branch points of M , and e is the number of end points of M :

- | | |
|-------------------------------------|-----------------------------|
| (a) $c = 0, b \leq 2, e$ is finite. | (e) $c = 2, e \leq 1$. |
| (b) $c = 0, b = 3, e = 5$. | (f) $c = 3, e = 0$. |
| (c) $c = 1, b \leq 1, e$ is finite. | (g) $c = 4, b = 1, e = 0$. |
| (d) $c = 1, b = e, e \leq 3$. | |

Proof. By virtue of the preceding remarks, it will be sufficient to show that in each case there exists a continuous (1—1) correspondence V , such that $V(M) = M'$ and such that sides are preserved under V .

For each set of values of c, b, e satisfying the conditions of the theorem, we shall describe a collection $[M_x]$ of point sets, such that any continuous curve having the given values of c, b and e can be put into correspondence with one and only one of the sets of this collection. Then corresponding to each set M_x , we shall describe a collection of sets $M_{x,1}, M_{x,2}, \dots, M_{x,k}$ lying in a plane S'' , and such that if M is a given continuous curve which is in continuous (1—1) correspondence with M_x and lies in a plane S , then there is one and only one set in this collection, say $M_{x,i}$, which is such that M and $M_{x,i}$ are in the same interior class with respect to S and S'' . It will be obvious in each case that if M and $M_{x,i}$ are in the same interior class, there exists a continuous (1—1) correspondence W , such that $W(M) = M_{x,i}$ and such that sides are preserved under W . If M and M' are in the same interior class with respect to S and S' , then M' and $M_{x,i}$ will be in the same interior class with respect to S' and S'' . Hence there exists a similar continuous (1—1) correspondence W' such that $W'(M_{x,i}) = M'$, and such that sides are preserved under W' . The correspondence $W'W$ is the required correspondence V mentioned above.

We shall have need of these relations in the following discussion: if $c = 0$, then $e \geq b + 2$; if $c = 1$, then $e \geq b$; if $c = 2$, then $e + 2 \geq b \geq 1$; if $c = 3$, then $e + 4 \geq b \geq 1$. Note also that if $c = 0$, condition (1) of the theorem is superfluous.

Case (a1). If $c = b = 0$, then $e = 2$, and the set $M_1 = M_{1,1}$ is an arc.

Case (a2). If $c = 0, b = 1, e = n$, the set $M_2 = M_{2,1}$ consists of n arcs, each having the point B for one end point, and no two of them having any other point in common.

Case (a3). If $c = 0, b = 2, e = n$, the set $M_3 = M_{3,1}$ consists of (1) an arc joining the points B_1 and B_2 ; (2) a collection of i arcs, where

$2 \leq i \leq n-2$, each arc having the point B_1 and no other point in common with the arc B_1B_2 , and no two of them having any other point in common; and (3) a collection of $n-i$ arcs each having the point B_2 and no other point in common with B_1B_2 , no two of them having any other point in common, and no one of them having a point in common with any arc of the other collection.

Case (b). If $c=0$, $b=3$, $e=5$, the set $M_4=M_{4,1}$ consists of four arcs E_1B_1 , E_2B_2 , E_3B_3 , E_4E_5 which are such that B_1 , B_2 , B_3 are interior points of the arc E_4E_5 , and except for these three points, no two of these arcs have any point in common.

Case (c1). If $c=1$, $b=0$, then $e=0$, and the set $M_5=M_{5,1}$ is a simple closed curve.

Case (c2). If $c=1$, $b=1$, $e=n$, the set M_6 consists of a simple closed curve C , and n arcs E_1B , E_2B , \dots , E_nB , each having the point B and no other point in common with C , and no two of them having any other point in common. The sets $M_{6,1}$, $M_{6,2}$, \dots , $M_{6,n+1}$ are defined thus: in the set $M_{6,i}$, there are $i-1$ end points interior to C , and $n-i+1$ end points exterior to C .

Case (d1). The case $c=1$, $b=e=0$ is the same as (c1).

Case (d2). The case $c=1$, $b=e=1$ is a special case of (c2).

Case (d3). For $c=1$, $b=e=2$, there are two sets: M_7 and M_8 . The set M_7 consists of a simple closed curve C and two arcs E_1B_1 , E_2B_2 having no point in common and having respectively the point B_1 , B_2 and no other point in common with C . In the set $M_{7,1}$ both end points are exterior to C ; in the set $M_{7,2}$ one is interior and one exterior to C ; in the set $M_{7,3}$ both are interior to C .

The set M_8 consists of (1) three arcs E_1B_2 , E_2B_2 , B_1B_2 , no two of which have any point in common except B_2 , and (2) a simple closed curve C which contains B_1 , but which has no other point in common with any one of the three arcs. In the set $M_{8,1}$ the point B_2 is exterior to C ; in the set $M_{8,2}$ it is interior to C .

Case (d4). For $c=1$, $b=e=3$, there are three sets: M_9 , M_{10} , M_{11} . The set M_9 consists of the set M_7 and an arc E_3B_3 having B_3 in common with $C-B_1-B_2$ and having no other point in common with M_7 . The sets $M_{9,1}$, \dots , $M_{9,4}$ are defined thus: in the set $M_{9,i}$ there are $i-1$ end points interior to C , and $4-i$ exterior to C .

The set M_{10} consists of the set M_8 and an arc E_3B_3 having B_3 in common with $C - B_1$ and having no other point in common with M_8 . The sets $M_{10,1}, \dots, M_{10,4}$ are defined in the same way as $M_{9,1}, \dots, M_{9,4}$ were defined.

The set M_{11} consists of the set M_8 and an arc E_3B_3 having B_3 and no other point in common with M_8 , where B_3 is an interior point of the arc B_1B_2 . The sets $M_{11,1}$ and $M_{11,2}$ are defined in the same way as $M_{8,1}$ and $M_{8,2}$ were defined.

Case (e1). If $c = 2, b = 1, e = 0$, the set M_{12} consists of two simple closed curves C_1, C_2 having the point B_1 and no other point in common. In the set $M_{12,1}, C_2$ is exterior to C_1 ; in the set $M_{12,2}, C_2$ is interior to C_1 .

Case (e2). If $c = 2, b = 2, e = 0$, the set M_{13} consists of two simple closed curves C_1, C_2 having no point in common, and an arc B_1B_2 having B_1 in common with C_1 and B_2 in common with C_2 , and having no other point in common with either of the simple closed curves. The sets $M_{13,1}$ and $M_{13,2}$ are defined in the same way as $M_{12,1}$ and $M_{12,2}$ were defined.

Case (e3). If $c = 2, b = 1, e = 1$, the set M_{14} consists of the set M_{12} and an arc E_1B_1 having no point in common with M_{12} except B_1 . In the set $M_{14,1}, C_1$ is exterior to C_2 , and E_1 is exterior to both C_1 and C_2 ; in $M_{14,2}, C_1$ is exterior to C_2 , and E_1 is interior to C_1 ; in $M_{14,3}, C_2$ is interior to C_1 , and E_1 is exterior to C_1 ; in $M_{14,4}, E_1$ and C_1 are interior to C_2 , and E_1 is exterior to C_1 ; in $M_{14,5}, E_1$ is interior to C_1 , and C_1 is interior to C_2 .

Case (e4). For $c = 2, b = 2, e = 1$, there are two sets: M_{15} and M_{16} . The set M_{15} consists of the set M_{12} and an arc E_1B_2 having B_2 in common with $C_1 - B_1$ and having no other point in common with M_{12} . The sets $M_{15,1}, \dots, M_{15,5}$ are defined in the same way as $M_{14,1}, \dots, M_{14,5}$ were defined; in the set $M_{15,6}, C_2$ and E_1 are interior to C_1 .

The set M_{16} consists of the set M_{13} and an arc E_1B_1 having B_1 and no other point in common with M_{13} . The sets $M_{16,1}, \dots, M_{16,6}$ are defined in the same way as $M_{15,1}, \dots, M_{15,6}$ were defined.

Case (e5). For $c = 2, b = 3, e = 1$, there are two sets: M_{17} and M_{18} . The set M_{17} consists of the set M_{13} and an arc E_1B_3 having B_3 in common with $C_1 - B_1$ and having no other point in common with M_{13} . The sets $M_{17,1}, \dots, M_{17,6}$ are defined in the same way as $M_{15,1}, \dots, M_{15,6}$ were defined.

The set M_{18} consists of the set M_{13} and an arc E_1B_3 having B_3 in common with $B_1B_2 - B_1 - B_2$ and having no other point in common with M_{13} . The

sets $M_{18,1}$ and $M_{18,2}$ are defined in the same way as $M_{12,1}$ and $M_{12,2}$ were defined.

Case (f1). If $c = 3$, $b = 1$, $e = 0$, the set M_{19} consists of three simple closed curves C_1 , C_2 , C_3 each containing the point B_1 and no two of them having any other point in common. In the set $M_{19,1}$, each simple closed curve is exterior to each of the other two; in $M_{19,2}$, C_2 and C_3 are interior to C_1 and exterior to each other; in $M_{19,3}$, C_1 is interior to C_2 , and C_2 is exterior to C_3 ; in $M_{19,4}$, C_1 is interior to C_2 , and C_2 is interior to C_3 .

Case (f2). For $c = 3$, $b = 2$, $e = 0$, there are three sets: M_{20} , M_{21} , M_{22} . The set M_{20} consists of three arcs joining the same pair of points, no two of the arcs having any interior points in common. Any two plane point sets which are in continuous (1 — 1) correspondence with M_{20} are in the same interior class with respect to the planes in which they lie; hence $M_{20,1} = M_{20}$.

The set M_{21} consists of the set M_{12} and a simple closed curve C_3 having the point B_2 in common with $C_2 - B_1$ and having no other point in common with M_{12} . The sets $M_{21,1}$, \dots , $M_{21,4}$ are defined in the same way as $M_{19,1}$, \dots , $M_{19,4}$ were defined; in the set $M_{21,5}$, C_1 and C_3 are interior to C_2 and exterior to each other.

The set M_{22} consists of the set M_{13} and a simple closed curve C_3 having B_2 and no other point in common with M_{13} . The sets $M_{22,1}$, \dots , $M_{22,5}$ are defined in the same way as $M_{21,1}$, \dots , $M_{21,5}$ were defined; in the set $M_{22,6}$, C_3 is interior to C_2 , and C_2 is exterior to C_1 ; in the set $M_{22,7}$, C_3 is interior to C_2 , and C_2 is interior to C_1 .

Case (f3). For $c = 3$, $b = 3$, $e = 0$, there are two sets: M_{23} and M_{24} . The set M_{23} consists of the set M_{13} and a simple closed curve having the point B_3 in common with $C_2 - B_2$ and having no other point in common with M_{13} . The sets $M_{23,1}$, \dots , $M_{23,7}$ are defined in the same way as $M_{22,1}$, \dots , $M_{22,7}$ were defined; in the set $M_{23,8}$, C_1 and C_2 are interior to C_3 and exterior to each other.

The set M_{24} consists of (1) the set M_{13} ; (2) a simple closed curve C_3 having no point in common with M_{13} ; and (3) an arc B_2B_3 having B_2 and no other point in common with M_{13} , and having B_3 and no other point in common with C_3 . The sets $M_{24,1}$, \dots , $M_{24,5}$ are defined in the same way as $M_{21,1}$, \dots , $M_{21,5}$ were defined.

Case (f4). For $c = 3$, $b = 4$, $e = 0$, there are two sets: M_{25} and M_{26} . The set M_{25} consists of (1) the set M_{13} ; (2) a simple closed curve C_3 having

no point in common with M_{13} ; and (3) an arc B_3B_4 having B_3 and no other point in common with C_3 , having B_4 in common with $B_1B_2 - B_1 - B_2$, and having no other point in common with M_{13} . The sets $M_{25,1}$ and $M_{25,2}$ are defined in the same way as $M_{19,1}$ and $M_{19,2}$ were defined.

The set M_{26} consists of (1) the set M_{13} ; (2) a simple closed curve C_3 having no point in common with M_{13} ; and (3) an arc B_3B_4 having B_3 and no other point in common with C_3 , having B_4 in common with $C_2 - B_2$, and having no other point in common with M_{13} . The sets $M_{26,1}, \dots, M_{26,5}$ are defined in the same way as $M_{21,1}, \dots, M_{21,5}$ were defined.

Case (g). If $c = 4, b = 1, e = 0$, the set M_{27} consists of four simple closed curves C_1, C_2, C_3, C_4 each containing the point B_1 , and no two of them having any other point in common. In the set $M_{27,1}$, each simple closed curve is exterior to each of the other three; in $M_{27,2}$, C_1 is interior to C_2 , and each of the simple closed curves C_2, C_3, C_4 is exterior to each of the other two; in $M_{27,3}$, C_1 is interior to C_2 , C_2 is exterior to C_4 , and C_3 is interior to C_4 ; in $M_{27,4}$, C_1 and C_2 are interior to C_3 and exterior to each other, and C_4 is exterior to C_3 ; in $M_{27,5}$, C_1 is interior to C_2 , C_2 is interior to C_3 , C_3 is exterior to C_4 ; in $M_{27,6}$, C_1, C_2, C_3 are each interior to C_4 , and each one is exterior to each one of the other two; in $M_{27,7}$, C_1 is interior to C_2 , C_3 is exterior to C_2 , C_2 and C_3 are interior to C_4 ; in $M_{27,8}$, C_1 and C_2 are interior to C_3 and exterior to each other, C_3 is interior to C_4 ; in $M_{27,9}$, C_1 is interior to C_2 , C_2 is interior to C_3 , C_3 is interior to C_4 .

This completes the proof of Theorem 1. We shall now show that the hypotheses of this theorem cannot be further weakened and the conclusion remain true. We give below a collection of examples in each of which are given two plane continuous curves M and M' , which are in the same interior class, and for which there exists a continuous $(1-1)$ correspondence T such that $T(M) = M'$. In each case c, b, e do not satisfy any one of the conditions (2) of the theorem and T cannot be A -extended.

Example 1. For $c = 0, b = 1, e$ infinite, the set M consists of the straight line intervals from $(0, 0)$ to $(1, 0)$, and from $(0, 0)$ to $(1/n, 1/n^2)$, for $n = 1, 2, 3, \dots$, and the set M' consists of the straight line intervals from $(0, 0)$ to $(1, 0)$, and from $(0, 0)$ to $[1/n, (-1)^n/n^2]$, for $n = 1, 2, 3, \dots$.

Example 2. For $c = 0, b = 2, e$ infinite, M and M' are the same sets as in Example 1 with the addition to each of the straight line interval from $(1/2, 0)$ to $(1, 1/2)$.

Example 3. For $c = 0$, $b = 3$, $e = 6$, the set M consists of the straight line intervals from $(0, 0)$ to $(4, 0)$, from $(1, 0)$ to $(1, 1)$, from $(2, 0)$ to $(2, 1)$, from $(2, 0)$ to $(5/2, 1)$, from $(3, 0)$ to $(3, 1)$, and M' is the same set except that the interval from $(2, 0)$ to $(5/2, 1)$ is replaced by the interval from $(2, 0)$ to $(2, -1)$.

Example 4. For $c = 0$, $b = 4$, $e = 6$, the set M consists of the straight line intervals from $(0, 0)$ to $(5, 0)$, and from $(n, 0)$ to $(n, 1)$ for $n = 1, 2, 3, 4$, and M' is the same set except that the interval from $(3, 0)$ to $(3, 1)$ is replaced by the interval from $(3, 0)$ to $(3, -1)$.

Example 5. For $c = 1$, $b = 1$, e infinite, M and M' are the same sets as in Example 1, with the addition to each of the circle $(x + 1)^2 + y^2 = 1$.

Example 6. For $c = 1$, $b = 2$, $e = 3$, the set M consists of the circle $x^2 + y^2 = 4$, and the straight line intervals from $(0, 1)$ to $(0, 2)$, and from $(1, 0)$ to $(3, 0)$, and M' consists of the same circle and the intervals from $(0, 2)$ to $(0, 3)$, from $(1, 0)$ to $(2, 0)$, and from $(1, 1)$ to $(2, 0)$.

Example 7. For $c = 1$, $b = 3$, $e = 4$, the sets M and M' are the same sets as in Example 6 with the addition to each of the interval from $(3/2, 0)$ to $(3/2, 1/4)$.

Example 8. For $c = 1$, $b = 4$, $e = 4$, the sets M and M' are the same sets as in Example 4 with the addition to each of the interval from $(0, 0)$ to $(1, 1)$.

Example 9. For $c = 2$, $b = 1$, $e = 2$, the set M consists of the straight line intervals from $(-1, 0)$ to $(1, 0)$, from $(0, -1)$ to $(0, 1)$, from $(-1, -1)$ to $(1, 1)$, from $(-1, -1)$ to $(-1, 0)$, and from $(1, 0)$ to $(1, 1)$, and M' is the same set except that the interval from $(1, 0)$ to $(1, 1)$ is replaced by the interval from $(0, 1)$ to $(1, 1)$.

Example 10. For $c = 3$, $b = 1$, $e = 1$, the set M consists of the straight line intervals from $(0, 0)$ to $(0, 4)$, from $(0, 4)$ to $(7, 4)$, from $(7, 4)$ to $(7, 0)$, from $(7, 0)$ to $(0, 0)$, from $(0, 0)$ to $(n, 1)$, for $n = 1, 2, 3, 4, 5$, from $(1, 1)$ to $(2, 1)$, and from $(3, 1)$ to $(4, 1)$, and M' is the same set except that the interval from $(3, 1)$ to $(4, 1)$ is replaced by the interval from $(4, 1)$ to $(5, 1)$.

Example 11. For $c = 4$, $b = 1$, $e = 1$, the sets M and M' are the same sets as in Example 10 with the addition to each of the circle $(x + 1)^2 + y^2 = 1$.

Example 12. For $c = 4$, $b = 2$, $e = 0$, the sets M and M' are the same sets as in Example 10 with the addition to M of the circle $(x-5)^2 + (y-2)^2 = 1$, and with the addition to M' of the circle $(x-3)^2 + (y-2)^2 = 1$.

Example 13. For $c = 5$, $b = 1$, $e = 0$, the set M consists of the set M of Example 10 with the addition of the straight line intervals from $(0, 0)$ to $(6, 1)$, from $(5, 1)$ to $(6, 1)$, from $(0, 0)$ to $(2/3, 1/2)$, from $(2/3, 1/2)$ to $(5/6, 1/2)$, and from $(5/6, 1/2)$ to $(0, 0)$, and M' is the same set except that the last three intervals are replaced by the intervals from $(0, 0)$ to $(5/3, 1/2)$, from $(5/3, 1/2)$ to $(11/6, 1/2)$, and from $(11/6, 1/2)$ to $(0, 0)$.

* * * * *

The second object of this paper is to prove a theorem which includes as special cases two of Antoine's theorems which occur in sections 22 and 23 of his paper.

DEFINITION. We shall say that the plane point set M is *reversible*, if given a simple closed curve J_0 enclosing M and a finite collection of simple closed curves J_1, J_2, \dots, J_n each one exterior to each of the others and each lying in a bounded domain complementary to M , there exists a continuous $(1-1)$ correspondence V , such that (1) $V(D) = D$, where D denotes that portion of the plane which is interior to J_0 and exterior to each of the simple closed curves J_1, J_2, \dots, J_n ; (2) $V(M) = M$; (3) $V(J_i) = J_i$, for $i = 0, 1, 2, \dots, n$; and (4) if A, B, C are three points on the simple closed curve J_0 , the sense ABC on J_0 is opposite to the sense $V(A)V(B)V(C)$ on J_0 .

LEMMA A. *If the plane point set M is reversible, then (5) if D, E, F are three points on the simple closed curve J_i ($i = 0, 1, \dots, n$) the sense DEF on J_i is opposite to the sense $V(D)V(E)V(F)$ on J_i .*

Of the plane continuous curves described in the proof of Theorem 1, all are reversible with the exception of the sets with the following subscripts: 11, 1 and 11, 2; 14, 1 and 14, 4; 16, 1 and 16, 4 and 16, 6; 18, 1 and 18, 2; 19, 1 and 19, 2; 22, 1 and 22, 2 and 22, 5; 24, 1 and 24, 2 and 24, 5; 25, 1 and 25, 2; 27, 1; 27, 2; 27, 4; 27, 6; 27, 7 and 27, 8. We can therefore state the following lemma:

LEMMA B. *The plane continuous curve M is reversible if c, b, e satisfy one of the following conditions:*

- | | |
|---|---------------------------------------|
| (a) $c = 0$, $b \leq 2$, e is finite. | (d') $c = 1$, $b = e$, $e \leq 2$. |
| (b) $c = 0$, $b = 3$, $e = 5$. | (e') $c = 2$, $e = 0$. |
| (c) $c = 1$, $b \leq 1$, e is finite. | |

THEOREM 2. *Given (1) a point set M lying in a plane S , and a continuous (1—1) correspondence T , such that $T(M) = M'$, where M' lies in a plane S' ; (2) each component* of M is bounded and except for at most one component, each is reversible; (3) if C denotes a simple closed curve in either S or S' , C encloses points of at most a finite number of components of either M or M' ; (4) for each component M_i of M , the correspondence T between M_i and the component $T(M_i) = M'_i$ of M' can be extended to a correspondence between the planes S and S' ; and (5) if M_i and M_j denote any two components of M , then M_i lies in the domain of $S - M_j$, bounded by the subset B of M_j , if and only if $T(M_i) = M'_i$ lies in the domain of $S' - M'_j$ bounded by $T(B)$. Under these conditions, the correspondence T between M and M' can be A -extended to a correspondence between the planes S and S' .*

Proof. The components of M are countable, by (3). Let us denote them by M_1, M_2, M_3, \dots . Then the components of M' are M'_1, M'_2, M'_3, \dots , where $T(M_i) = M'_i$. Let us suppose them so numbered that for $i > 1$, M_i and M'_i are reversible.

By (4), there exists corresponding to each component M_i of M , a continuous (1—1) correspondence U_i , such that $U_i(S) = S'$, and for each point P of M_i , $U_i(P) = T(P)$. Hence $U_i(M_i) = T(M_i) = M'_i$.

By (3), no point of M or of M' is a limit point of the set consisting of all components which do not contain the given point. Hence, corresponding to each component M_i , we can construct a simple closed curve J_i in S , such that (a) J_i has no points in common with M or with $J_1 + J_2 + \dots + J_{i-1}$; (b) J_i encloses M_i but encloses no other components of M save those which lie in a bounded domain of $S - M_i$; (c) the simple closed curve $U_i(J_i) = J'_i$ in S' has no points in common with M' or with $J'_1 + J'_2 + \dots + J'_{i-1}$; and (d) J'_i encloses M'_i but encloses no other components of M' save those which lie in a bounded domain of $S' - M'_i$.

By (3), each simple closed curve J_i encloses at most a finite number of simple closed curves of the collection $J_1 + J_2 + \dots$. Of all those which are enclosed by J_i , let $J_{i_1}, J_{i_2}, \dots, J_{i_x}$ denote those which are not interior to any other simple closed curve enclosed by J_i . Then $J_i + J_{i_1} + J_{i_2} + \dots + J_{i_x}$ form the boundary of a bounded domain D_i which is complementary to the set $J_1 + J_2 + \dots$. The simple closed curve J_i is the outer boundary of D_i ; each of the simple closed curves J_{i_n} ($n = 1, 2, \dots, x$) is a component of the inner boundary of D_i . The domain D_i contains M_i but

* A component (= maximal connected subset) of a point set M is a connected subset of M which is not a proper subset of any other connected subset of M .

contains no points of $M - M_i$. Similarly in S' , J'_i is the outer boundary, and $J'_{i_1} + J'_{i_2} + \dots + J'_{i_x}$ the inner boundary of a domain D'_i , which contains M'_i but contains no points of $M' - M'_i$.

In order to prove the theorem we must define a continuous (1—1) correspondence U such that $U(S) = S'$ and $U(M) = M'$. We shall do this in such a way that for each component M_i of M , $U(M_i) = T(M_i) = M'_i$. We shall first define U for points of D_1 and its boundary.

Let d_1, d_2, \dots, d_{k_1} denote those domains of $S - M_1$ that contain components of the inner boundary of D_1 . By (5), $U_1(d_i) = d'_i$ is a domain of $S' - M'_1$ which contains a component of the inner boundary of D'_1 . Now let V_1 be a continuous (1—1) correspondence such that (a) $V_1(P) = U_1(P)$ for all points P of $S - (d_1 + d_2 + \dots + d_{k_1})$, and hence $V_1(M_1) = M'_1$ and $V_1(J_1) = J'_1$; (b) $V_1(d_i) = d'_i$, for $i = 1, 2, \dots, k_1$, and hence $V_1(S) = S'$; and (c) $V_1(J_{1n}) = J'_{1n}$, for $n = 1, 2, \dots, x_1$, and hence $V_1(D_1) = D'_1$. We shall let $U(P) = V_1(P)$ for all points of D_1 and its boundary.

Let D_a be a domain such that U has been defined for points of J_a . Let A, B, C be three points of J_a . If the sense $U(A)U(B)U(C)$ is the same as the sense $U_a(A)U_a(B)U_a(C)$ on the simple closed curve J'_a , there exists a continuous (1—1) correspondence V_a such that (a) $V_a(S) = S'$; (b) for each point P of J_a , $V_a(P) = U(P)$, and hence $V_a(J_a) = J'_a$; (c) for each point P of M_a , $V_a(P) = U_a(P)$, and hence $V_a(M_a) = T(M_a) = M'_a$; and (d) $V_a(J_{an}) = J'_{an}$, for $n = 1, 2, \dots, x_a$, and hence $V_a(D_a) = D'_a$. Since M_a is reversible, even if the two senses are opposite, there exists a correspondence V_a such that (a), (b) and (d) are true, and (c) is replaced by (c') $V_a(M_a) = T(M_a) = M'_a$. In both cases, we shall let $U(P) = V_a(P)$ for all points of D_a and its boundary.

Let D_b be a domain such that U has been defined for points of the component J_{b_1} of its inner boundary. Here again there exists a continuous (1—1) correspondence V_b such that (a) for each point P of J_{b_1} , $V_b(P) = U(P)$; (b) $V_b(M_b) = T(M_b) = M'_b$; (c) $V_b(J_b) = J'_b$; (d) $V_b(J_{b_n}) = J'_{b_n}$, for $n = 2, 3, \dots, x_b$; and (e) $V_b(D_b) = D'_b$. We shall let $U(P) = V_b(P)$ for all points of D_b and its boundary.

Having defined U for points of D_1 , we can by the process outlined in the two preceding paragraphs, define U for all points of S interior to J_1 , and for all points of S interior to a simple closed curve J_c which encloses J_1 . If this includes all points of S , the theorem has been proved. In case this does not include all points of S , we shall construct in S an infinite collection of simple closed curves C_1, C_2, C_3, \dots , such that (a) C_i is interior to C_{i+1} ;

(b) C_i has no points in common with M ; (c) for each point P of S there is some integer k such that P is interior to C_k . In S' , we shall construct an infinite collection of simple closed curves C'_1, C'_2, C'_3, \dots , such that (a) C'_i is interior to C'_{i+1} ; (b) C'_i has no points in common with M' ; (c) for each point P of S' there is some integer k such that P is interior to C'_k ; (d) C'_i encloses a component M'_i of M' , if and only if C_i encloses M_i . Let W_i ($i=1, 2, 3, \dots$) be a continuous $(1-1)$ correspondence such that $W_i(C_i) = C'_i$. If we let $N = M + C_1 + C_2 + \dots$ and $N' = M' + C'_1 + C'_2 + \dots$ and let V be a continuous $(1-1)$ correspondence such that for each point P of M , $V(P) = T(P)$, and for each point P of C_i , $V(P) = W_i(P)$, then N, N', V satisfy the conditions of Theorem 2. If then we order the components of N , construct a new collection of simple closed curves J_1, J_2, J_3, \dots , and proceed as outlined above, a correspondence U will be defined for all points of S in such a way that $U(S) = S'$ and $U(M) = M'$, which proves Theorem 2.

YALE UNIVERSITY.

Green's Functions for Differential Systems Containing a Parameter

BY W. W. ELLIOTT.

1. *Introduction.* This paper is a continuation of the paper entitled 'Generalized Green's Functions for Compatible Differential Systems,' which appeared in Vol. 50 of this Journal, pp. 243-258. The notations and equation numbers of the former paper are continued in this paper, and references are made accordingly. The ϕ_i 's and the ψ_i 's are respectively assumed to form biorthogonal sets with the u_i 's and the v_i 's as suggested at the close of the above named paper.

2. *Relations Between Differential and Integral Systems Containing a Parameter.* Let us now consider in connection with the compatible differential system (1), the system

$$(29) \quad L(u) = \lambda u, \quad U_p(u) = 0, \quad (p = 1, \dots, n),$$

where λ is a parameter ranging over the complex plane.

The system adjoint to (29) will be

$$(30) \quad M(v) = \lambda v, \quad V_p(v) = 0, \quad (p = 1, \dots, n).$$

In general systems (29) and (30) are compatible for some values of λ and incompatible for others. However, in special cases systems (29) and (30) are incompatible for all values of λ . Such systems will possess ordinary Green's functions for all parameter values. Cases also arise in which systems (29) and (30) are compatible for all* values of λ . Such systems possess generalized Green's functions for all parameter values. In ordinary cases, systems (29) and (30) will possess ordinary Green's functions for some parameter values and generalized Green's functions for other parameter values. If systems (29) and (30) are compatible of order h for some value of λ , their solutions will be respectively denoted by $u_1^\lambda, \dots, u_h^\lambda$, and $v_1^\lambda, \dots, v_h^\lambda$. The ϕ_i 's and ψ_i 's will also be given a superscript to denote the value of λ for which they are taken.

The generalized Green's functions will be given two superscripts. The first one will be the value of λ , and the second will denote the order of com-

* M. Bôcher, *Les Méthodes de Sturm*, p. 42.

patibility of the given system for this value of λ . In this notation G with a second superscript zero will mean an ordinary Green's function, while with the second superscript different from zero it will mean a generalized Green's function. The assumption as to what parameter value we are considering, the order of compatibility for this value, and the indication whether the Green's function considered is the ordinary or generalized one will then be implied from this notation without the necessity of explicit statement in each case.

At this point several natural choices of the ϕ_i^λ 's and ψ_i^λ 's may be presented.

The ϕ_i^λ 's and ψ_i^λ 's may be taken respectively as sets of functions linearly equivalent to the conjugates of the sets of solutions u_i^λ 's and v_i^λ 's. The Green's function resulting from this choice of the ϕ_i^λ 's and ψ_i^λ 's will be given the subscript 1.

Some of the following work will simplify if a set of ϕ_i^λ 's can be found which, in addition to forming a biorthogonal set with the u_i^λ 's, are orthogonal to each u^μ , ($\mu \neq \lambda$), and if a set of ψ_i^λ 's can be found which, in addition to forming a biorthogonal set with the v_i^λ 's, are orthogonal to each v^μ , ($\mu \neq \lambda$). The Green's functions resulting from this choice of the ϕ_i^λ 's and ψ_i^λ 's, when they exist, will be given the subscript zero.

Provided each u^λ multiplied by v^λ and integrated from a to b does not give zero, the above condition will be satisfied by the ϕ_i^λ 's and ψ_i^λ 's, if the ϕ_i^λ 's are chosen linearly equivalent to the v_i^λ 's and the ψ_i^λ 's to the u_i^λ 's. The Green's function resulting from this further special choice, when it exists, will be given the double subscript 01.

It should be noted that the functions $G_1(xy)$ and $G_{01}(xy)$ depend solely upon the given differential system and are independent of any particular choice of the complete sets of solutions.

The function $G_{01}(xy)$ always exists for Hermitian differential systems. In fact for Hermitian differential systems $G_{01}(xy) = G_1(xy)$. In dealing with Hermitian differential systems in the future we shall always use the special function $G_1(xy)$, as it will serve to give all the facts. For other systems, so far as possible, the general Green's function without special choice of the ϕ_i^λ 's and ψ_i^λ 's will be used. The results for special choices of the ϕ_i^λ 's and ψ_i^λ 's will be pointed out specifically only when they are more simple than in the general case.

If zero is taken as one of the chosen parameter values some of the work simplifies. An assumption about the differential system for $\lambda = 0$ is essen-

tially equivalent to an assumption about it for any fixed parameter value, since we can always effect a linear transformation on the parameter.

From Theorem V and Cor. II readily follows

THEOREM VI. *If the differential system is Hermitian and $n > 1$, then*

$$G_1^{\lambda k}(xy) \equiv G_1^{\lambda k}(yx).$$

THEOREM VII. *Solutions of the integral equation*

$$(31) \quad u(x) = \lambda \int_a^b [G^{0k}(xs) + (1/\lambda) \sum_{i=1}^k u_i^0(x) \phi_i^0(s)] u(s) ds,$$

are linearly equivalent to those of the system

$$(32) \quad \begin{aligned} L(u) &= \lambda u(x) - \lambda \sum_{i=1}^k \psi_i^0(x) \int_a^b v_i^0(s) u(s) ds, \\ U_p(u) &= 0, \quad (p = 1, \dots, n). \end{aligned}$$

That any solution of (31) satisfies (32) is seen at once by applying the differential operator L and the boundary conditions to equation (31). To show that any solution of (32) satisfies (31) apply Green's theorem, letting $u = u(x)$, any solution of (32) which has a continuous n^{th} derivative, $v = H^{0k}(xy)$, and simplify.

THEOREM VIII. *Solutions of the equation*

$$(33) \quad \theta(x) = \lambda \int_a^b [G^{0k}(xs) + (1/\lambda) \sum_{i=1}^k u_i^0(x) \phi_i^0(s)] \theta(s) ds,$$

are expressible linearly in terms of those of the system

$$(34) \quad \begin{aligned} \theta_i(x) &= u_i^0(x) + \lambda \int_a^b G^{0k}(xs) \theta_i(s) ds, \quad (i = 1, \dots, k), \\ \theta_0(x) &= \lambda \int_a^b G^{0k}(xs) \theta_0(s) ds. \end{aligned}$$

By way of proof, equation (33) may be written in the form

$$(35) \quad \theta(x) = \lambda \int_a^b G^{0k}(xs) \theta(s) ds + \sum_{i=1}^k c_i u_i^0(x),$$

where $c_i = \int_a^b \phi_i^0(s) \theta(s) ds$, ($i = 1, \dots, k$). Each c_i is given in terms of θ , but if we write equation (35) for arbitrary c_i it follows, on multiplying by $\theta_i^0(x)$, ($i = 1, \dots, k$), and integrating with respect to x from a to b , that

$$c_i = \int_a^b \phi_i^0(s) \theta(s) ds, \quad (i = 1, \dots, k),$$

and (35) is satisfied. Therefore it suffices to consider (35) for arbitrary c_i . Assign to c_i , ($i = 1, \dots, k$), the k sets of values

$$\begin{aligned} &1, 0, 0, \dots, 0, \\ &0, 1, 0, \dots, 0, \\ &0, 0, 1, \dots, 0, \\ &\dots \dots \dots \dots \dots \dots \dots \\ &0, 0, 0, \dots, 1. \end{aligned}$$

Upon substituting these sets of values in (35) the following equations are obtained

$$(36) \quad \theta_i(x) = u_i^0(x) + \lambda \int_a^b G^{0k}(xs) \theta_i(s) ds, \quad (i = 1, \dots, k).$$

Now let $\theta(x)$ be any solution of (35) and form the difference

$$(37) \quad \theta_0(s) = \theta(s) - \sum_{i=1}^k c_i \theta_i(s).$$

On multiplying (37) by $\lambda G^{0k}(xs)$ and integrating with respect to s from a to b , we get

$$\begin{aligned} \lambda \int_a^b G^{0k}(xs) \theta_0(s) ds &= \lambda \int_a^b G^{0k}(xs) \theta(s) ds - \lambda \sum_{i=1}^k c_i \int_a^b G^{0k}(xs) \theta_i(s) ds \\ &= \theta(x) - \sum_{i=1}^k c_i \theta_i(x) \\ &= \theta_0(x). \end{aligned}$$

This shows that $\theta_0(x)$ is a solution of the equation

$$(38) \quad \theta_0(x) = \lambda \int_a^b G^{0k}(xs) \theta_0(s) ds.$$

Then a complete set of solutions of (35) may be obtained by adding solutions of equation (38) to those of system (36).

It is obvious that solutions of (36) and (38) satisfy (35) for some set of c_i 's.

COROLLARY I. *The equation*

$$\theta(x) = \lambda \int_a^b [G^{0k}(xs) + (1/\lambda) \sum_{i=1}^k u_i^0(x) \phi_i^0(s)] \theta(s) ds,$$

always has a solution not identically zero.

For at values of λ which are roots of the determinant function * $D(\lambda)$

* T. Lalesco, *Equations Intégrales*, p. 21.

formed for $G^{0k}(xs)$, (38) has solutions not identically zero, and at all other values of λ , equations (36) have solutions.

THEOREM IX. *Solutions of the system*

$$(39) \quad \begin{aligned} L(u) &= \lambda u - \lambda \sum_{i=1}^k \psi_i^0(x) \int_a^b v_i^0(s) u(s) ds, \\ U_p(u) &= 0, \end{aligned} \quad (p=1, \dots, n),$$

are expressible linearly in terms of those of the systems

$$(40) \quad \begin{aligned} L(u) &= \lambda u(x) - \lambda \psi_i^0(x), & (i=1, \dots, k), \\ U_p(u) &= 0, & (p=1, \dots, n), \end{aligned}$$

and

$$(41) \quad L(u_0) = \lambda u_0, \quad U_p(u_0) = 0, \quad (p=1, \dots, n).$$

The first equation of system (39) may be written

$$(42) \quad L(u) = \lambda u - \lambda \sum_{i=1}^k c_i \psi_i^0(x),$$

Where $c_i = \int_a^b v_i^0(s) u(s) ds$. Here again the c_i 's are given in terms of the function u , but if equation (42) is written for arbitrary c_i 's it follows on application of Green's theorem that

$$(43) \quad c_i = \int_a^b v_i^0(s) u(s) ds, \quad (i=1, \dots, k),$$

so that equation (42) is satisfied. The remainder of the proof is similar to that of theorem VIII.

From theorems VII, VIII, and IX follows a rather symmetrical result which may be stated as a theorem.

THEOREM X. *A complete set of solutions of the integral system*

$$\begin{aligned} \theta_i(x) &= u_i^0(x) + \lambda \int_a^b G^{0k}(xs) \theta_i(s) ds, & (i=1, \dots, k), \\ \theta_0(x) &= \lambda \int_a^b G^{0k}(xs) \theta_0(s) ds, \end{aligned}$$

taken as a whole, is linearly equivalent to a complete set of solutions of the differential systems

$$\begin{aligned} L(u) &= \lambda u - \lambda \psi_i^0, & U_p(u) &= 0, & (p=1, \dots, n; \quad i=1, \dots, k), \\ \text{and} \\ L(u_0) &= \lambda u_0, & U_p(u_0) &= 0, & (p=1, \dots, n), \end{aligned}$$

taken as a whole.

THEOREM XI. *Solutions of the equation*

$$(44) \quad \theta(x) = \lambda \int_a^b G^{0k}(xs) \theta(s) ds,$$

are linearly equivalent to those of the system

$$(45) \quad \begin{aligned} L(u) &= \lambda u(x) - \lambda \sum_{i=1}^k \psi_i^0(x) \int_a^b v_i^0(s) u(s) ds, \\ U_p(u) &= 0, \quad (p = 1, \dots, n), \\ \int_a^b \phi_i^0(x) u(x) dx &= 0, \quad (i = 1, \dots, k). \end{aligned}$$

That any solution of (44) satisfies system (45) is seen at once by performing the indicated operations on the function defined by (44). To show that any solution of system (45) satisfies (44) apply Green's theorem, letting u = any solution of (45) and $v = H^{0k}(xy)$.

THEOREM XII. *Solutions of the integral system*

$$(46) \quad \begin{aligned} \theta(x) &= \lambda \int_a^b G^{0k}(xs) \theta(s) ds + \sum_{i=1}^k u_i^0(x) \int_a^b \phi_i^0(s) \theta(s) ds, \\ \int_a^b v_i^0(s) \theta(s) ds &= 0, \quad (i = 1, \dots, k), \end{aligned}$$

are linearly equivalent to those of the differential system

$$(47) \quad L(u) = \lambda u, \quad U_p(u) = 0, \quad (p = 1, \dots, n).$$

That every solution of (46) satisfies (47) is seen by performing the indicated operations on the function $\theta(x)$ and taking into account the condition $\int_a^b v_i^0(s) \theta(s) ds = 0, \quad (i = 1, \dots, k).$

To show that every solution of (47) satisfies system (46) apply Green's theorem, letting $u = u(x)$, any solution of (47), and $v = H^{0k}(xy)$. In the simplification use is made of the fact that every solution of (47) satisfies the condition

$$\int_a^b v_i^0(s) u(s) ds = 0 \quad (i = 1, \dots, k).$$

If the Green's function $G_{01}^{0k}(xy)$ exists for the given system, we have the simple equivalence theorem:

THEOREM XIII. *If the function $G_{01}^{0k}(xy)$ exists* for the given differential system, then solutions of the integral equation*

* This hypothesis is equivalent to assuming that G_{01}^{0k} exists for some value of λ .

$$\theta(x) = \lambda \int_a^b G_{01}^{0k}(xs) \theta(s) ds, \quad \lambda \neq 0,$$

are equivalent to those of the differential system

$$L(u) = \lambda u, \quad \lambda \neq 0, \quad U_p(u) = 0, \quad (p = 1, \dots, n).$$

Since for a Hermitian system G_{01}^{0k} always exists and equals G_1^{0k} we may state:

THEOREM XIV. *For a given Hermitian differential system the solutions of the integral equation*

$$(48) \quad \theta(x) = \lambda \int_a^b G_1^{0k}(xs) \theta(s) ds, \quad \lambda \neq 0,$$

are equivalent to those of the differential system

$$(49) \quad L(u) = \lambda u, \quad U_p(u) = 0, \quad (p = 1, \dots, n). \quad \lambda \neq 0.$$

Now suppose the function $D(\lambda)$ is formed for the kernel $G_1^{0k}(xs)$ of a Hermitian system and the roots of $D(\lambda)$ are found to be $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \dots$. If we choose a real number λ' not equal to 0, $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \dots$, and define

$$\Gamma_1^{0k}(xs) = G_1^{0k}(xs) + (1/\lambda') \sum_{i=1}^k u_i^0(x) \bar{u}_i^0(s),$$

we may state

THEOREM XV. *Solutions of the equation*

$$(50) \quad \theta(x) = \lambda \int_a^b \Gamma_1^{0k}(xs) \theta(s) ds$$

are equivalent to those of the Hermitian system

$$(51) \quad L(u) = \lambda u, \quad U_p(u) = 0, \quad (p = 1, \dots, n),$$

the value $\lambda = 0$ not being excluded.

Since

$$\begin{aligned} \theta(x) &= \lambda \int_a^b \Gamma_1^{0k}(xs) \theta(s) ds \\ &= \lambda \int_a^b [G_1^{0k}(xs) + (1/\lambda') \sum_{i=1}^k u_i^0(x) \bar{u}_i^0(s)] \theta(s) ds, \end{aligned}$$

and

$$\int_a^b \bar{u}_p^0(x) \theta(x) dx = (\lambda/\lambda') \int_a^b \bar{u}_p^0(s) \theta(s) ds, \quad (p = 1, \dots, k),$$

$$\text{either } \lambda = \lambda', \text{ or } \int_a^b \bar{u}_p^0(s) \theta(s) ds = 0, \quad (p = 1, \dots, k).$$

If $\int_a^b \bar{u}_p^0(s)\theta(s)ds = 0$, then $\theta(x) = \lambda \int_a^b G_1^{0k}(xs)\theta(s)ds$, and λ must be one of the values $\lambda_1, \lambda_2, \lambda_3, \dots$; and solutions of equation (50) must be linearly equivalent to those of equation (48).

If $\lambda = \lambda'$

$$\theta(x) = \lambda' \int_a^b G_1^{0k}(xs)\theta(s)ds + \sum_{i=1}^k u_i^0(x) \int_a^b \bar{u}_i^0(s)\theta(s)ds.$$

Let

$$\alpha(s) = \theta(s) - \sum_{i=1}^k u_i^0(s) \int_a^b \bar{u}_i^0(t)\theta(t)dt.$$

Multiplying by $\lambda' G_1^{0k}(xs)$ and integrating from a to b , we get

$$\begin{aligned} \lambda' \int_a^b G_1^{0k}(xs)\alpha(s)ds &= \lambda' \int_a^b G_1^{0k}(xs)\theta(s)ds \\ &\quad - \lambda' \sum_{i=1}^k \int_a^b \int_a^b G_1^{0k}(xs)u_i^0(s)\bar{u}_i^0(t)\theta(t)dt ds \\ &= \lambda' \int_a^b G_1^{0k}(xs)\theta(s)ds \\ &= \theta(x) - \sum_{i=1}^k u_i^0(x) \int_a^b \bar{u}_i^0(t)\theta(t)dt \\ &= \alpha(x). \end{aligned}$$

Since $\lambda' \neq \lambda_1, \lambda_2, \lambda_3, \dots$, $\alpha(x) = 0$, and $\theta(x) = \sum_{i=1}^k c_i u_i^0(x)$. Hence solutions of (50) are linearly equivalent to those of (48) plus linear combinations of $u_i^0(x)$, ($i = 1, \dots, k$), and theorem (XV) follows from theorem (XIV).

THEOREM XVI. *The kernel $\Gamma_1^{0k}(xs)$ is a closed* kernel.*

Suppose $\int_a^b \Gamma_1^{0k}(xs)\beta(s)ds = 0$. This becomes, on substituting for $\Gamma_1^{0k}(xs)$, $\int_a^b G_1^{0k}(xs)\beta(s)ds + (1/\lambda') \sum_{j=1}^k u_j^0(x) \int_a^b \bar{u}_j^0(s)\beta(s)ds = 0$.

Multiplying by $\bar{u}_i^0(x)$, ($i = 1, \dots, k$), and integrating, we get

$$\int_a^b \bar{u}_i^0(s)\beta(s)ds = 0, \quad (i = 1, \dots, k),$$

and then

$$\int_a^b G_1^{0k}(xs)\beta(s)ds = 0.$$

* D. Hilbert, *Göttinger Nachrichten* (1904), p. 49.

Applying the differential operator L to this equation gives

$$(52) \quad \beta(x) \equiv \sum_{i=1}^k u_i^0(x) \int_a^b \bar{u}_i^0(s) \beta(s) ds \equiv 0,$$

which shows the kernel is closed.

From theorems XIV, XV, XVI, together with the theory of integral equations with Hermitian kernels, follows *

THEOREM XVII. *A Hermitian differential system cannot be compatible for all values of the parameter λ . The values of λ for which the system is compatible are infinite in number, are all real, and have no finite limit point.*

3. *Relations between Green's functions for different values of the parameter.*

THEOREM XVIII. *In systems for which $G^{\lambda h}(xy)$ and $G^{00}(xy)$ both exist, $G^{\lambda h}(xy)$ is a pseudoresolvent † function to the function $G^{00}(xy)$.*

Apply Green's theorem, ‡ letting

$$U = G^{\lambda h}(x\xi), \text{ and } v = H^{00}(x\eta).$$

This gives after a simplification and a change of variables

$$(53) \quad G^{\lambda h}(xy) = G^{00}(xy) + \lambda \int_a^b G^{00}(xs) G^{\lambda h}(sy) ds \\ - \int_a^b G^{00}(xs) \sum_{i=1}^h \psi_i^\lambda(s) v_i^\lambda(y) ds,$$

or

$$(54) \quad G^{\lambda h}(xy) = G^{00}(xy) + \lambda \int_a^b G^{00}(xs) G^{\lambda h}(sy) ds - (1/\lambda) \sum_{i=1}^h \Psi_i^\lambda(x) v_i^\lambda(y),$$

where

$$(55) \quad \Psi_i^\lambda(x) = \lambda \int_a^b G^{00}(xs) \psi_i^\lambda(s) ds, \quad (i = 1, \dots, h).$$

Similarly, by letting $u = G^{00}(x\xi)$, $v = H^{\lambda h}(x\eta)$, applying Green's theorem, and simplifying, we get

$$(56) \quad G^{\lambda h}(xy) = G^{00}(xy) + \lambda \int_a^b G^{\lambda h}(xs) G^{00}(sy) ds - (1/\lambda) \sum_{i=1}^h \Phi_i^\lambda(y) u_i^\lambda(x),$$

* This result has been previously shown; see Bôcher, *Proceedings of the Fifth International Congress of Mathematicians*, Vol. 1, p. 185.

† W. A. Hurwitz, *Transactions of the American Mathematical Society*, Vol. 13 (1912), p. 405.

‡ See theorem V.

where

$$(57) \quad \Phi_i^\lambda(y) = \lambda \int_a^b \phi_i^\lambda(s) G^{00}(sy) ds, \quad (i = 1, \dots, k).$$

From the theory of the ordinary* Green's function it is known that the u_i^λ 's and v_i^λ 's form complete sets of linearly independent solutions of the equations

$$u^\lambda(x) = \lambda \int_a^b G^{00}(xs) u^\lambda(s) ds,$$

$$v^\lambda(x) = \lambda \int_a^b H^{00}(xs) v^\lambda(s) ds.$$

Then

$$\begin{aligned} \int_a^b u_i^\lambda(x) \Phi_j^\lambda(x) dx &= \lambda \int_a^b \int_a^b u_i^\lambda(x) G^{00}(sx) \phi_j^\lambda(s) ds dx \\ &= \int_a^b \phi_j^\lambda(s) u_i^\lambda(s) ds = \begin{cases} 0, & i \neq j, \\ 1, & i = j, \end{cases} \quad (i, j = 1, \dots, h). \end{aligned}$$

Similarly

$$\int_a^b v_i^\lambda(x) \Psi_j^\lambda(x) dx = \begin{cases} 0, & i \neq j, \\ 1, & i = j, \end{cases} \quad (i, j = 1, \dots, h).$$

Thus the necessary and sufficient conditions† that $G^{\lambda h}(xy)$ be a pseudo resolvent to $G^{00}(xy)$ are satisfied.

The function $G^{\lambda h}(xy)$ is not uniquely determined by equations (54) and (56). However, it may be completely determined by demanding in addition one of the relations,

$$\begin{aligned} \int_a^b \Phi_j^\lambda(x) G^{\lambda h}(xy) dx &= (1/\lambda) \left[\sum_{i=1}^h v_i^\lambda(y) \int_a^b \Phi_j^\lambda(s) \psi_i^\lambda(s) ds - \Phi_j^\lambda(y) \right], \\ \int_a^b \Psi_j^\lambda(x) G^{\lambda h}(xy) dx &= (1/\lambda) \left[\sum_{i=1}^h u_i^\lambda(y) \int_a^b \Psi_j^\lambda(s) \phi_i^\lambda(s) ds - \Psi_j^\lambda(y) \right]. \end{aligned}$$

That these relations are satisfied by the function $G^{\lambda h}(xy)$ is obtained by applications of Green's theorem.

The last theorem simplifies somewhat if the Green's function $G_{01}^{\lambda h}(xy)$ is used. In this case the functions Ψ_j^λ and Φ_j^λ are respectively equivalent to the functions u_j^λ and v_j^λ , and we may state

THEOREM XIX. *In systems for which $G_{01}^{\lambda h}(xy)$ and $G^{00}(xy)$ both exist, $G_{01}^{\lambda h}(xy)$ is that pseudoresolvent function to $G^{00}(xy)$ which satisfies the conditions,*

* M. Bôcher, *Les Méthodes de Sturm*, p. 109.

† W. A. Hurwitz, *loc. cit.*

$$G_{01}^{\lambda h}(xy) = G^{00}(xy) + \lambda \int_a^b G^{00}(xs) G_{01}^{\lambda h}(sy) ds \\ - (1/\lambda) \sum_{i=1}^h u_i^{\lambda}(x) v_i^{\lambda}(y),$$

$$G_{01}^{\lambda h}(xy) = G^{00}(xy) + \lambda \int_a^b G_{01}^{\lambda h}(xs) G^{00}(sy) ds \\ - (1/\lambda) \sum_{i=1}^h v_i^{\lambda}(y) u_i^{\lambda}(x),$$

$$\int_a^b v_j^{\lambda}(x) G_{01}^{\lambda h}(xy) dx = 0, \quad (j = 1, \dots, k),$$

$$\int_a^b u_j^{\lambda}(x) G_{01}^{\lambda h}(yx) dx = 0, \quad (j = 1, \dots, k).$$

COROLLARY I. In a Hermitian differential system, $G_1^{\lambda h}(xy)$ is that pseudoresolvent function to the kernel $G^{00}(xy)$ which satisfies the conditions

$$G_1^{\lambda h}(xy) = G^{00}(xy) + \lambda \int_a^b G^{00}(xs) G_1^{\lambda h}(sy) ds \\ - (1/\lambda) \sum_{i=1}^h \bar{v}_i^{\lambda}(x) v_i^{\lambda}(y),$$

$$G_1^{\lambda h}(xy) = G^{00}(xy) + \lambda \int_a^b G_1^{\lambda h}(xs) G^{00}(sy) ds \\ - (1/\lambda) \sum_{i=1}^h \bar{u}_i^{\lambda}(y) u_i^{\lambda}(x),$$

$$\int_a^b \bar{u}_i^{\lambda}(x) G_1^{\lambda h}(xy) dx = 0,$$

$$\int_a^b \bar{v}_i^{\lambda}(x) G_1^{\lambda h}(yx) dx = 0.$$

The functions $\Psi_i^{\lambda}(x)$ and $\Phi_i^{\lambda}(x)$ defined by equations (55) and (57) are respectively identical with the solutions of the differential systems,

$$(58) \quad L[\Psi_i^{\lambda}(x)] = \lambda \psi_i^{\lambda}(x), \quad U_p[\Psi_i^{\lambda}(x)] = 0, \\ (p = 1, \dots, n; i = 1, \dots, k).$$

and

$$(59) \quad M[\Phi_i^{\lambda}(x)] = \lambda \phi_i^{\lambda}(x), \quad V_p[\Phi_i^{\lambda}(x)] = 0, \\ (p = 1, \dots, n; i = 1, \dots, k).$$

For each i the solution of each of these systems is unique. The Φ_i^{λ} 's and the Ψ_i^{λ} 's are then completely determined by the differential systems (58) and (59).

Define the functions

$$Q^{\lambda h}(x\xi) = G^{\lambda h}(x\xi) - (1/\lambda) \sum_{i=1}^h \sum_{j=1}^h c^{\lambda}_{ij} u_i^{\lambda}(x) v_j^{\lambda}(\xi) \\ + (1/\lambda) \sum_{i=1}^h u_i^{\lambda}(x) v_i^{\lambda}(\xi),$$

$$Q^{00}(x\eta) = G^{00}(x\eta) - (1/\lambda) \sum_{i=1}^h \Psi_i^{\lambda}(x) \Phi_i^{\lambda}(\eta),$$

$$R^{\lambda h}(x\xi) = H^{\lambda h}(x\xi) - (1/\lambda) \sum_{i=1}^h \sum_{j=1}^h b^{\lambda}_{ij} v_i^{\lambda}(x) u_j^{\lambda}(\xi) \\ + (1/\lambda) \sum_{i=1}^h v_i^{\lambda}(x) u_i^{\lambda}(\xi),$$

$$R^{00}(x\eta) = H^{00}(x\eta) - (1/\lambda) \sum_{i=1}^h \Phi_i^{\lambda}(x) \Psi_i^{\lambda}(\eta),$$

where

$$c^{\lambda}_{ij} = \int_a^b \Phi_i^{\lambda}(v) \psi_j^{\lambda}(x) dx, \quad b^{\lambda}_{ij} = \int_a^b \Psi_i^{\lambda}(x) \phi_j^{\lambda}(x) dx.$$

Then $Q^{00}(x\eta) = R^{00}(\eta x)$, and also, since $c^{\lambda}_{ij} = b^{\lambda}_{ji}$, $Q^{\lambda h}(x\xi) = R^{\lambda h}(\xi x)$.

THEOREM XX. *The function $Q^{\lambda h}(xy)$ is the resolvent function to $Q^{00}(xy)$ for the parameter value λ .*

Apply Green's Theorem, letting

$$u = Q^{\lambda h}(x\xi), \text{ and } v = R^{00}(x\eta).$$

This gives, after simplification involving somewhat lengthy calculations and a change of variables,

$$(60) \quad Q^{\lambda h}(xy) = Q^{00}(xy) + \lambda \int_a^b Q^{00}(xs) Q^{\lambda h}(sy) ds.$$

Similarly, by letting $u = Q^{00}(x\xi)$, $v = R^{\lambda h}(x\eta)$, and applying Green's theorem, we have

$$(61) \quad Q^{\lambda h}(xy) = Q^{00}(xy) + \lambda \int_a^b Q^{\lambda h}(xs) Q^{00}(sy) ds.$$

Equations (60) and (61) determine the resolvent relationship between the functions $Q^{\lambda h}(xy)$ and $Q^{00}(xy)$.

In systems for which the more specialized function $G_{01}^{\lambda h}$ exists, we may state

THEOREM XXI. *The function $G_{01}^{\lambda h}(xy)$, if it exists, is the resolvent function to $Q^{00}(xy)$ where*

$$Q^{00}(xy) = G^{00}(xy) - (1/\lambda) \sum_{i=1}^h u_i^{\lambda}(x) v_i^{\lambda}(y).$$

COROLLARY. For a Hermitian differential system $G_1^{\lambda\lambda}(xy)$ is the resolvent function to $G^{00}(xy) - (1/\lambda) \sum_{i=1}^h u_i^{\lambda}(x)v_i^{\lambda}(y)$.

It remains to find the relations between the generalized Green's functions for two different values of the parameter λ , each of which renders the system compatible. Here again it would be desirable to find a resolvent relationship between the Green's functions and combinations of other functions, at the two parameter values, but this seems rather difficult in the general case and a relationship of a different kind has been found. The resolvent relationship has been found for the more special function $G_0(xy)$.

Define the functions,

$$(62) \quad \Psi_j^{\lambda}(x) = \lambda \int_a^b G^{0k}(xs) \psi_j^{\lambda}(s) ds, \quad (j = 1, \dots, h),$$

$$(63) \quad \Psi_j^0(x) = \lambda \int_a^b G^{\lambda h}(xs) \psi_j^0(s) ds, \quad (j = 1, \dots, k),$$

$$(64) \quad \Phi_j^{\lambda}(x) = \lambda \int_a^b H^{0k}(xs) \phi_j^{\lambda}(s) ds, \quad (j = 1, \dots, h),$$

$$(65) \quad \Phi_j^0(x) = \lambda \int_a^b H^{\lambda h}(xs) \phi_j^0(s) ds, \quad (j = 1, \dots, k).$$

For each j there is a unique function defined by each of these equations, and this function is identical with the unique solution given, for the same j , in the corresponding systems,

$$(62a) \quad \begin{aligned} L[\Psi_j^{\lambda}(x)] &= \lambda \psi_j^{\lambda}(x) - \lambda \sum_{i=1}^k \psi_i^0(x) \int_a^b v_i^0(s) \psi_j^{\lambda}(s) ds, \\ U_p[\Psi_j^{\lambda}(x)] &= 0, \quad (p = 1, \dots, n), \\ \int_a^b \phi_i^0(x) \Psi_j^{\lambda}(x) dx &= 0, \quad (i = 1, \dots, k; j = 1, \dots, h); \end{aligned}$$

$$(63a) \quad \begin{aligned} L[\Psi_j^0(x)] &= \lambda \Psi_j^0(x) + \lambda \psi_j^0(x) - \lambda \sum_{i=1}^h \psi_i^{\lambda}(x) \int_a^b v_i^{\lambda}(s) \psi_j^0(s) ds, \\ U_p[\Psi_j^0(x)] &= 0, \quad (p = 1, \dots, n), \\ \int_a^b \phi_i^{\lambda}(x) \Psi_j^0(x) dx &= 0, \quad (i = 1, \dots, h; j = 1, \dots, k); \end{aligned}$$

$$(64a) \quad \begin{aligned} M[\Phi_j^{\lambda}(x)] &= \lambda \phi_j^{\lambda}(x) - \lambda \sum_{i=1}^k \phi_i^0(x) \int_a^b u_i^0(s) \phi_j^{\lambda}(s) ds, \\ V_p[\Phi_j^{\lambda}(x)] &= 0, \quad (p = 1, \dots, n), \\ \int_a^b \psi_i^0(x) \Phi_j^{\lambda}(x) dx &= 0, \quad (i = 1, \dots, k; j = 1, \dots, h); \end{aligned}$$

$$\begin{aligned}
 M[\Phi_j^0(x)] &= \lambda \Phi_j^0(x) + \lambda \phi_j^0(x) - \lambda \sum_{i=1}^h \phi_i^\lambda(x) \int_a^b u_i^\lambda(s) \phi_i^0(s) ds, \\
 (65a) \quad V_p[\Phi_j^0(x)] &= 0, \quad (p=1, \dots, n), \\
 \int_a^b \psi_i^\lambda(x) \Phi_j^0(x) dx &= 0, \quad (i=1, \dots, h; j=1, \dots, k).
 \end{aligned}$$

THEOREM XXII. In systems for which $G^{0k}(xy)$ and $G^{\lambda h}(xy)$ both exist, and $G^{0k}(xy)$ is known, $G^\lambda(xy)$ may be determined either by a solution of the equation

$$\begin{aligned}
 (66) \quad G^{\lambda h}(xy) &= G^{0k}(xy) - (1/\lambda) \sum_{i=1}^h v_i^\lambda(y) \Psi_i^\lambda(x) \\
 &\quad + (1/\lambda) \sum_{i=1}^k u_i^0(x) \Phi_i^0(y) + \lambda \int_a^b G^{0k}(xs) G^{\lambda h}(sy) ds,^*
 \end{aligned}$$

which satisfies the conditions

$$(67) \quad \int_a^b v_j^0(x) G^{\lambda h}(xy) dx = (1/\lambda) \left[\sum_{i=1}^h v_i^\lambda(y) \int_a^b v_i^0(s) \psi_i^\lambda(s) ds - v_j^0(y) \right],$$

($j=1, \dots, k$),

$$(68) \quad \int_a^b \phi_j^\lambda(x) G^{\lambda h}(xy) dx = 0, \quad (j=1, \dots, h),$$

or by a solution of the equation

$$\begin{aligned}
 (66a) \quad G^{\lambda h}(xy) &= G^{0k}(xy) - (1/\lambda) \sum_{i=1}^h u_i^\lambda(y) \Phi_i^\lambda(x) \\
 &\quad + (1/\lambda) \sum_{i=1}^k v_i^0(x) \Psi_i^0(y) + \lambda \int_a^b G^{\lambda h}(xs) G^{0k}(sy) ds,
 \end{aligned}$$

which satisfies the conditions

$$(67a) \quad \int_a^b u_j^0(x) G^{\lambda h}(yx) dx = (1/\lambda) \left[\sum_{i=1}^h u_i^\lambda(y) \int_a^b u_i^0(s) \phi_i^\lambda(s) ds - u_j^0(y) \right],$$

($j=1, \dots, k$),

$$(68a) \quad \int_a^b \psi_j^\lambda(x) G^{\lambda h}(yx) dx = 0.$$

That a solution of equations (66), (67) and (68) exists may be shown indirectly by showing they are satisfied by the Green's function $G^{\lambda h}(xy)$. It may be seen by application of Green's theorem that (66) and (67) are satisfied by $G^{\lambda h}(xy)$. Condition (68) follows from the definition of $G^{\lambda h}(xy)$.

But a solution of (66), (67) and (68) is unique; for suppose there are two solutions $G^{\lambda h}(xy)$ and $G_2^{\lambda h}(xy)$, and form the difference

$$R(xy) = G^{\lambda h}(xy) - G_2^{\lambda h}(xy).$$

The function $R(xy)$ will satisfy the system

* The Φ_i 's and Ψ_i 's are defined by the systems (62a), (63a), (64a) and (65a).

$$\begin{aligned}
 R(xy) &= \lambda \int_a^b G^{0k}(xs) R(sy) ds, \\
 (69) \quad \int_a^b v_j^0(x) R(xy) dx &= 0, & (j=1, \dots, k), \\
 \int_a^b \phi_j^\lambda(x) R(xy) dx &= 0, & (j=1, \dots, h).
 \end{aligned}$$

Every solution of system (69) satisfies the system

$$\begin{aligned}
 (a) \quad L_x[R(xy)] &= \lambda R(xy), \\
 (70) \quad (b) \quad U_p[R(xy)] &= 0, & (p=1, \dots, n), \\
 (c) \quad \int_a^b \phi_j^\lambda(x) R(xy) dx &= 0, & (j=1, \dots, h).
 \end{aligned}$$

But there are no solutions of (70) which do not vanish identically, for every solution of (70a, b) is given by

$$R(xy) = \sum_{i=1}^h c_i(y) u_i^\lambda(x),$$

and from (70c), $c_i(y) \equiv 0$, ($i=1, \dots, h$). Hence

$$R(xy) \equiv 0, \text{ and } G^{\lambda h}(xy) \equiv G_2^{\lambda h}(xy).$$

It may be shown similarly that the Green's function $G^{\lambda h}(xy)$ may be obtained from the alternative equations (66a), (67a) and (68a).

The statement of theorem XXII does not simplify to a great extent for the functions $G_0^{0k}(xy)$ and $G_0^{\lambda h}(xy)$, but for the functions $G_{01}^{0k}(xy)$ and $G_{01}^{\lambda h}(xy)$ it is more symmetric, and it will be restated.

THEOREM XXIII. *In systems for which $G_{01}^{0k}(xy)$ and $G_{01}^{\lambda h}(xy)$ both exist, and G_{01}^{0k} is known, $G_{01}^{\lambda h}(xy)$ may be obtained, either as the solution of equations*

$$\begin{aligned}
 G_{01}^{\lambda h}(xy) &= G_{01}^{0k}(xy) - (1/\lambda) \sum_{i=1}^h v_i^\lambda(y) u_i^\lambda(x) \\
 &\quad - (1/\lambda) \sum_{i=1}^k u_i^0(x) v_i^0(y) + \lambda \int_a^b G_{01}^{0k}(xs) G_{01}^{\lambda h}(sy) ds, \\
 \int_a^b v_j^0(x) G_{01}^{\lambda h}(xy) dx &= -(1/\lambda) v_j^0(y), & (j=1, \dots, k), \\
 \int_a^b v_j^\lambda(x) G_{01}^{\lambda h}(xy) dx &= 0, & (j=1, \dots, h),
 \end{aligned}$$

or the solution of the equations

$$\begin{aligned}
 G_{01}^{\lambda h}(xy) &= G_{01}^{0k}(xy) - (1/\lambda) \sum_{i=1}^h u_i^\lambda(y) v_i^\lambda(x) \\
 &\quad - (1/\lambda) \sum_{i=1}^k v_i^0(x) u_i^0(y) + \lambda \int_a^b G_{01}^{0k}(xs) G_{01}^{\lambda h}(sy) ds, \\
 \int_a^b u_j^0(x) G_{01}^{\lambda h}(yx) dx &= -(1/\lambda) u_j^0(y), & (j=1, \dots, k), \\
 \int_a^b u_j^\lambda(x) G_{01}^{\lambda h}(yx) dx &= 0, & (j=1, \dots, h).
 \end{aligned}$$

If the function $G_{01}(xy)$ is used, we should have, instead of the functions defined by (62a), (63a), (64a), and (65a), those defined by the following systems

$$(71) \quad \begin{aligned} L[\Psi^{\lambda}_{0j}(x)] &= \lambda \psi_j^{\lambda}(x), & U_p[\Psi^{\lambda}_{0j}(x)] &= 0, \\ \int_a^b \phi_i^0(x) \Psi^{\lambda}_{0j}(x) dx &= 0, & (i=1, \dots, k; j=1, \dots, h; \\ & & p=1, \dots, n); \end{aligned}$$

$$(72) \quad \begin{aligned} L[\Psi^0_{0j}(x)] &= \lambda \Psi^0_{0j}(x) + \lambda \psi_j^0(x), & U_p[\Psi^0_{0j}(x)] &= 0, \\ \int_a^b \phi_i^{\lambda}(x) \Psi^0_{0j}(x) dx &= 0, & (i=1, \dots, h; j=1, \dots, k; \\ & & p=1, \dots, n); \end{aligned}$$

$$(73) \quad \begin{aligned} M[\Phi^{\lambda}_{0j}(x)] &= \lambda \phi_j^{\lambda}(x), & V_p[\Phi^{\lambda}_{0j}(x)] &= 0, \\ \int_a^b \psi_i^0(x) \Phi^{\lambda}_{0j}(x) dx &= 0, & (i=1, \dots, k; j=1, \dots, h; \\ & & p=1, \dots, n); \end{aligned}$$

$$(74) \quad \begin{aligned} M[\Phi^0_{0j}(x)] &= \lambda \Phi^0_{0j}(x) + \lambda \phi_j^0(x), & V_p[\Phi^0_{0j}(x)] &= 0, \\ \int_a^b \psi_i^{\lambda}(x) \Phi^0_{0j}(x) dx &= 0, & (i=1, \dots, h; j=1, \dots, k; \\ & & p=1, \dots, n). \end{aligned}$$

Define the functions

$$\begin{aligned} Q_0^{\lambda h}(xy) &= G_0^{\lambda h}(xy) + (1/\lambda) \sum_{i=1}^k \Psi^0_{0i}(x) \Phi^0_{0i}(y) \\ &\quad + (1/\lambda) \sum_{i=1}^h \sum_{j=1}^h c_{ij}^{\lambda} u_i^{\lambda}(x) v_j^{\lambda}(y) - (1/\lambda) \sum_{i=1}^h u_i^{\lambda}(x) v_i^{\lambda}(y), \\ Q_0^{0k}(xy) &= G_0^{0k}(xy) - (1/\lambda) \sum_{i=1}^h \Psi^0_{0i}(x) \Phi^0_{0i}(y) \\ &\quad + (1/\lambda) \sum_{i=1}^k \sum_{j=1}^k c^{\lambda}_{ij} u_i^{\lambda}(x) v_j^{\lambda}(y) - (1/\lambda) \sum_{i=1}^k u_i^0(x) v_i^0(y), \\ R_0^{\lambda h}(xy) &= H_0^{\lambda h}(xy) + (1/\lambda) \sum_{i=1}^k \Phi^0_{0i}(x) \Psi^0_{0i}(y) \\ &\quad + (1/\lambda) \sum_{i=1}^h \sum_{j=1}^h b_{ij}^{\lambda} v_i^{\lambda}(x) u_j^{\lambda}(y) - (1/\lambda) \sum_{i=1}^h v_i^{\lambda}(x) u_i^{\lambda}(y), \\ R_0^{0k}(xy) &= H_0^{0k}(xy) - (1/\lambda) \sum_{i=1}^h \Phi^{\lambda}_{0i}(x) \Psi^{\lambda}_{0i}(y) \\ &\quad + (1/\lambda) \sum_{i=1}^k \sum_{j=1}^k b^0_{ij} v_i^0(x) u_j^0(y) - (1/\lambda) \sum_{i=1}^k v_i^0(x) u_i^0(y), \end{aligned}$$

where the Φ_{0j} 's and Ψ_{0j} 's are the functions defined above and where

$$\begin{aligned} c^{\lambda}_{ij} &= \int_a^b \psi_j^{\lambda}(t) \Phi^{\lambda}_{0i}(t) dt, & c^0_{ij} &= \int_a^b \psi_j^0(t) \Phi^0_{0i}(t) dt, \\ b^{\lambda}_{ij} &= \int_a^b \phi_j^{\lambda}(t) \Psi^{\lambda}_{0i}(t) dt, & b^0_{ij} &= \int_a^b \phi_j^0(t) \Psi^0_{0i}(t) dt. \end{aligned}$$

It is readily seen that $Q_0^{\lambda h}(xy) = R_0^{\lambda h}(yx)$, and $Q_0^{0k}(xy) = R_0^{0k}(yx)$.

THEOREM XXIV. *In systems for which both functions exist, the function $Q_0^{\lambda h}(xy)$ is the resolvent function to $Q_0^{0k}(xy)$ for the parameter value λ .*

The proof follows from proper application of Green's theorem though the calculations are rather laborious.

For the more specialized Green's functions we may state

THEOREM XXV. *In systems for which both functions exist the function $Q_{01}^{\lambda h}(xy)$ is the resolvent function to $Q_{01}^{0k}(xy)$ for the parameter value λ , where $Q_{01}^{\lambda h}(xy)$ and $Q_{01}^{0k}(xy)$ are defined by*

$$Q_{01}^{\lambda h}(xy) = G_{01}^{\lambda h}(xy) - (1/\lambda) \sum_{p=1}^k u_p^0(x) v_p^0(y),$$

$$Q_{01}^{0k}(xy) = G_{01}^{0k}(xy) - (1/\lambda) \sum_{p=1}^h u_p^{\lambda}(x) v_p^{\lambda}(y).$$

COROLLARY. *For Hermitian differential systems*

$$G_1^{\lambda h}(xy) - (1/\lambda) \sum_{p=1}^k \bar{v}_p^0(x) v_p^0(y)$$

is the resolvent function to

$$G_1^{0k}(xy) - (1/\lambda) \sum_{p=1}^h \bar{v}_p^{\lambda}(x) v_p^{\lambda}(y)$$

for the parameter value λ .

4. Expansion Theorems. In this section, when solutions for more than one value of the parameter are considered, we shall denote by λ_i ($i = 1, 2, 3, \dots$), the values of λ which render the given differential system compatible, the subscripts being arranged in the order of magnitude of the absolute value of λ . The order of compatibility for each λ_i will be denoted by k_i , and the normalized solutions of the system for each λ_i will be given a superscript λ_i . In this notation, if the system is compatible for $\lambda = 0$, then $\lambda_0 = 0$.

If complex numbers are used, the Hilbert-Schmidt* expansion theorem for integral equations with Hermitian kernels states that every function of the form

$$g(x) = \int_a^b K(xs) h(s) ds,$$

where $h(s)$ is any continuous function and $K(xs)$ is a continuous Hermitian kernel, is developable in a series uniformly and absolutely convergent of the form

* E. Schmidt, *Mathematische Annalen*, Vol. 63 (1907), p. 433; E. Goursat, *Cours d'Analyse*, Vol. 3 (1915), p. 446.

$$g(x) = \sum_{\gamma} \phi_{\gamma}(x) \int_a^b \bar{\phi}_{\gamma}(s) g(s) ds,$$

the ϕ_{γ} 's being normalized solutions of the equation

$$\phi(x) = \lambda \int_a^b K(xs) \phi(s) ds.$$

From this follows readily,

THEOREM XXVI. *If the differential system (29) is Hermitian, any given function $f(x)$ having n continuous derivatives and satisfying $U_p[f(x)] = 0$, ($p = 1, \dots, n$), may be expanded in an absolutely and uniformly convergent series of the form*

$$\begin{aligned} f(x) &= \sum_{i=0}^{\infty} \sum_{p=1}^{k_i} u_p^{\lambda_i}(x) \int_a^b \bar{u}_p^{\lambda_i}(s) f(s) ds \\ &= \sum_{i=1}^{\infty} \sum_{p=1}^{k_i} \bar{v}_p^{\lambda_i}(x) \int_a^b v_p^{\lambda_i}(s) f(s) ds. \end{aligned}$$

In order to obtain this from the Hilbert-Schmidt theorem choose* $K(xs) = \Gamma_1^{0k_0}(xs)$. The ϕ_{γ} 's are then equivalent to the $u_p^{\lambda_i}$'s.

To show that any $f(x)$ which satisfies the conditions of the theorem may be written

$$f(x) = \int_a^b \Gamma_1^{0k_0}(xs) h(s) ds.$$

where $h(s)$ is a continuous function, apply Green's theorem letting $u = f(x)$ and $v = \Gamma_1^{0k_0}(sx)$. This gives, after simplification

$$f(x) = \int_a^b \Gamma_1^{0k_0}(xs) L[f(s)] ds + \sum_{j=1}^{k_0} u_j^0(x) \int_a^b \bar{u}_j^0(t) f(t) dt.$$

But

$$\begin{aligned} \lambda' \sum_{i=1}^{k_0} \int_a^b \int_a^b \Gamma_1^{0k_0}(xs) \bar{v}_i^0(s) \bar{u}_i^0(t) f(t) dt ds \\ = \sum_{i=1}^{k_0} \sum_{j=1}^{k_0} u_j^0(x) \int_a^b \int_a^b \bar{u}_j^0(s) \bar{v}_i^0(s) \bar{u}_i^0(t) f(t) dt ds \\ = \sum_{i=1}^{k_0} u_i^0(x) \int_a^b \bar{u}_i^0(t) f(t) dt. \end{aligned}$$

Then we may write

$$f(x) = \int_a^b \Gamma_1^{0k_0}(xs) \{L[f(s)] + \lambda' \sum_{i=1}^{k_0} u_i^0(s) \int_a^b \bar{u}_i^0(t) f(t) dt\} ds.$$

Hence, if

$$h(s) = L[f(s)] + \lambda' \sum_{i=1}^{k_0} u_i^0(s) \int_a^b \bar{u}_i^0(t) f(t) dt,$$

then

$$f(x) = \int_a^b \Gamma_1^{0k_0}(xs) h(s) ds.$$

* The function Γ_1 was defined on page 403.

From Theorem XVII there are an infinite number of λ , which render the system compatible.

LEMMA. *Solutions of the equations*

$$(75) \quad \phi(x) = \lambda \int_a^b G_1^{0k}(xs) \bar{\psi}(s) ds,$$

$$(76) \quad \psi(x) = \lambda \int_a^b H_1^{0k}(xs) \bar{\phi}(s) ds,$$

are equivalent to those of the systems

$$(77) \quad L[\phi(x)] = \lambda \bar{\psi}(x), \quad U_p[\phi(x)] = 0, \quad (p = 1, \dots, n),$$

and

$$(78) \quad M[\psi(x)] = \lambda \bar{\phi}(x), \quad V_p[\psi(x)] = 0, \quad (p = 1, \dots, n), \quad \lambda \neq 0.$$

In order to show that any solution of (75) satisfies (77) apply the indicated operation to (75). This gives

$$L[\phi(x)] = \lambda \bar{\psi}(x) - \sum_{i=1}^k \bar{v}_i^0(x) \int_a^b v_i^0(s) \bar{\psi}(s) ds,$$

$$U_p[\phi(x)] = 0, \quad (p = 1, \dots, n).$$

But from (76)

$$\int_a^b \bar{v}_i^0(s) \psi(s) ds = 0, \quad (i = 1, \dots, k).$$

Hence

$$L[\phi(x)] = \lambda \bar{\psi}(x), \quad U_p[\phi(x)] = 0, \quad (p = 1, \dots, n).$$

Similarly it may be shown that $\psi(x)$ of (76) satisfies (78).

The converse is shown by an application of Green's theorem.

E. Schmidt* also showed how to expand a function in terms of the fundamental functions, defined by him, for an integral equation with non-symmetric kernel. This result, written out for complex functions, would state that every function $g(x)$, which may be expressed in the form

$$g(x) = \int_a^b K(xs) h(s) ds,$$

where $K(xs)$ is any continuous function of x and s , and $h(s)$ is a continuous function, is developable in a uniformly and absolutely convergent series of the form

$$g(x) = \sum_i \sum_{p=1}^{k_i} \phi_p^{\lambda_i}(x) \int_a^b \bar{\phi}_p^{\lambda_i}(s) g(s) ds,$$

* E. Schmidt, *loc. cit.*; E. Goursat, *Cours d'Analyse*, Vol. 3 (1915), p. 470.

the $\phi_p^{\lambda_i}$'s being solutions of the system

$$(79) \quad \phi(x) = \lambda \int_a^b K(xs) \bar{\psi}(s) ds,$$

$$(80) \quad \psi(x) = \lambda \int_a^b K(sx) \bar{\phi}(s) ds.$$

From this theorem and the preceding lemma follows

THEOREM XXVII. *Any given function $f(x)$ having n continuous derivatives and satisfying $U_p[f(x)] = 0$, ($p = 1, \dots, n$), may be expanded in an absolutely and uniformly convergent series of the form*

$$f(x) = \sum_i \sum_{p=1}^{k_i} \phi_p^{\lambda_i}(x) \int_a^b \bar{\phi}_p^{\lambda_i}(s) f(s) ds,$$

where the $\phi_p^{\lambda_i}$'s are solutions of system (77).

To obtain this ϕ choose

$$K(xs) = G^{0k_0}(xs).$$

Then solutions of (79) and (80) are, by the lemma above, equivalent to those of (77) and (78) for $\lambda \neq 0$. Suppose we are given any $f(x)$ satisfying the conditions of the theorem. Form from this $f(x)$ the function

$$f_1(x) = f(x) - \sum_{j=1}^{k_0} u_j^0(x) \int_a^b \bar{u}_j^0(s) f(s) ds.$$

The function $f_1(x)$ satisfies the conditions of the theorem and also the condition

$$\int_a^b \bar{u}_j^0(x) f_1(x) dx = 0, \quad (j = 1, \dots, k).$$

To show that the function $f_1(x)$ may be expressed in the form

$$(81) \quad f_1(x) = \int_a^b G^{0k_0}(xs) h(s) ds,$$

where $h(s)$ is a continuous function, apply Green's theorem letting $u = f_1(x)$ and $v = G^{0k_0}(sx)$. After simplification, this gives $f_1(x)$ in form (81) where, $h(s) = L[f(s)]$. Then from Schmidt's theorem

$$f_1(x) = \sum_i \sum_{p=1}^{k_i} \phi_p^{\lambda_i}(x) \int_a^b \bar{\phi}_p^{\lambda_i}(s) f_1(s) ds, \quad i \neq 0.$$

Whence

$$\begin{aligned} f(x) &= \sum_{j=1}^{k_0} u_j^0(x) \int_a^b \bar{u}_j^0(s) f(s) ds \\ &\quad + \sum_i \sum_{p=1}^{k_i} \phi_p^{\lambda_i}(x) \int_a^b \bar{\phi}_p^{\lambda_i}(s) f(s) ds, \quad i \neq 0, \end{aligned}$$

or

$$f(x) = \sum_i \sum_{p=1}^{k_i} \phi_p^{\lambda_i}(x) \int_a^b \bar{\phi}_p^{\lambda_i}(s) f(s) ds,$$

where $i = 0$ is not excluded.

A Class of Invariant Functionals of Quadratic Functional Forms.*

BY T. S. PETERSON.

1. *Introduction.* Michal has shown † that the coefficient $g_{\alpha\beta}$ of the absolute quadratic functional form

$$(1.1) \quad g_{\alpha\beta} y^\alpha y^\beta + \int_a^b (y^\alpha)^2 d\alpha \quad (g_{\alpha\beta} = g_{\beta\alpha})$$

has the law of transformation

$$(1.2) \quad \bar{g}_{\alpha\beta} = g_{\alpha\beta} + g_{\lambda\beta} K_\alpha^\lambda + g_{\alpha\mu} K_\beta^\mu + g_{\lambda\mu} K_\alpha^\lambda K_\beta^\mu \\ + K_\beta^\alpha + K_\alpha^\beta + \int_a^b K_\alpha^\sigma K_\beta^\sigma d\sigma$$

under the Fredholm integral transformation

$$(1.3) \quad y^t = \bar{y}^t + K_\alpha^t \bar{y}^\alpha.$$

In the above formulae we have employed the extension to integration † of the Einstein summation notation. *This convention of letting the repetition of a continuous variable, once as a subscript and once as a superscript, in a term denote a Riemann integration with respect to this subscript over the interval (a, b) will be used throughout our paper.*

The study of invariant functionals of functional forms (1.1) was begun by Michal.† It is the purpose of this paper to develop still further the invariance properties of such functional forms.

2. *Functional transformations.* We shall first demonstrate the following theorem.

THEOREM I. *The law of transformation,*

$$(2.1) \quad \bar{g}_{\alpha\beta} = g_{\alpha\beta} + g_{\lambda\beta} K_\alpha^\lambda + g_{\alpha\mu} K_\beta^\mu + g_{\lambda\mu} K_\alpha^\lambda K_\beta^\mu \\ + K_\beta^\alpha + K_\alpha^\beta + \int_a^b K_\alpha^\sigma K_\beta^\sigma d\sigma$$

* Presented to the Society, April 6, 1928.

† Cf. A. D. Michal, "Affinely Connected Function Space Manifolds," *American Journal of Mathematics*, Vol. 50 (1928), pp. 473-517.

of the coefficient of the quadratic functional form

$$g_{\alpha\beta}y^\alpha y^\beta + \int_a^b (y^\alpha)^2 d\alpha,$$

which remains invariant under the Fredholm group of transformations (1.3) with non-vanishing Fredholm determinants, possesses the group property.

In order to prove the above theorem, let us consider two distinct members of the class of transformations (2.1) with kernels A_α^i , B_α^i , and denote them by T_1 , T_2 respectively. By taking the product $T_1 T_2$ of the transformations T_1 and T_2 , we obtain

$$(2.2) \quad \bar{g}_{\alpha\beta} = g_{\alpha\beta} + g_{\gamma\beta} Q_\alpha^\gamma + g_{\alpha\delta} Q_\beta^\delta + g_{\gamma\delta} Q_\alpha^\gamma Q_\beta^\delta \\ + Q_\beta^\alpha + Q_\alpha^\beta + \int_a^b Q_\alpha^\sigma Q_\beta^\sigma d\sigma$$

where $Q_\alpha^i = A_\alpha^i + B_\alpha^i + A_\sigma^i B_\alpha^\sigma$. This verifies the closure property.

To prove the existence of an inverse, we shall use a method of proof similar to that used by Volterra* in his treatment of equation (1.3). Let us substitute new variables in (2.1), multiply by k_α^i , and integrate in order to obtain the equations for the following terms: $\bar{g}_{\alpha\beta}$, $\bar{g}_{\alpha\mu} k_\beta^\mu$, $\bar{g}_{\lambda\beta} k_\alpha^\lambda$, and $\bar{g}_{\lambda\mu} k_\alpha^\lambda k_\beta^\mu$. If, to the sum of these four expressions we add $k_\beta^\alpha + k_\alpha^\beta + \int_a^b k_\alpha^\sigma k_\beta^\sigma d\sigma$ to both sides, we shall obtain

$$\bar{g}_{\alpha\beta} + \bar{g}_{\lambda\beta} k_\alpha^\lambda + \bar{g}_{\alpha\mu} k_\beta^\mu + \bar{g}_{\lambda\mu} k_\alpha^\lambda k_\beta^\mu \\ + k_\beta^\alpha + k_\alpha^\beta + \int_a^b k_\alpha^\sigma k_\beta^\sigma d\sigma = g_{\alpha\beta} + g_{\gamma\beta} P_\alpha^\gamma \\ + g_{\alpha\delta} P_\beta^\delta + g_{\gamma\delta} P_\alpha^\gamma P_\beta^\delta + P_\beta^\alpha + P_\alpha^\beta + \int_a^b P_\alpha^\sigma P_\beta^\sigma d\sigma$$

where $P_\alpha^i = K_\alpha^i + k_\alpha^i + K_\sigma^i k_\alpha^\sigma$. This however is the fundamental relation between a kernel and its resolvent kernel and is always zero. Thus we have established a continuous inverse

$$(2.3) \quad g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \bar{g}_{\lambda\beta} k_\alpha^\lambda + \bar{g}_{\alpha\mu} k_\beta^\mu + \bar{g}_{\lambda\mu} k_\alpha^\lambda k_\beta^\mu \\ + k_\beta^\alpha + k_\alpha^\beta + \int_a^b k_\alpha^\sigma k_\beta^\sigma d\sigma.$$

We see then, that under the conditions we have imposed, (2.1) cannot have more than one continuous solution; and, if it has one, this will be given by formula (2.3). In order to prove that the continuous function $g_{\alpha\beta}$ defined

* Cf. Bôcher, "Introduction to the Study of Integral Equations," Cambridge Tract No. 10, Cambridge (1926), pp. 21-22.

by (2.3) is really a solution of (2.1), one needs only to note that by substituting (2.3) in (2.1), we obtain an identity.

Obviously we shall have the identical transformation when the kernel K_a^i is equal to zero.

3. *Functional invariants.* A sequence of continuous functionals $f_{(i)\lambda\mu\cdots\nu}^{a\beta\cdots\gamma}[g_{\sigma\tau}]$, ($i=1, 2, \cdots, n$) of the coefficients $g_{\sigma\tau}$ of the quadratic functional form (1.1) that depends in particular on the variables $\alpha, \beta, \cdots, \gamma, \lambda, \mu, \cdots, \nu$ will be said to form an *invariant class of functionals** of weight $w(=\text{const.})$, if, under the Fredholm transformations (1.3) with non-vanishing Fredholm determinants, we have

$$(3.1) \quad f_{(i)\lambda\mu\cdots\nu}^{a\beta\cdots\gamma}[\bar{g}_{\sigma\tau}] \\ = (D[K_i^{\sigma}])^w \phi_{\lambda\mu\cdots\nu}^{a\beta\cdots\gamma} [f_{(1)lm\cdots n}^{ab\cdots c}, f_{(2)lm\cdots n}^{ab\cdots c}, \cdots, f_{(i)lm\cdots n}^{ab\cdots c}, K_q^p, k_v^u] \\ (i=1, 2, \cdots, n)$$

where $\phi_{\lambda\mu\cdots\nu}^{a\beta\cdots\gamma}$ is a fixed functional of its functional arguments $f_{(1)lm\cdots n}^{ab\cdots c}, f_{(2)lm\cdots n}^{ab\cdots c}, \cdots, f_{(i)lm\cdots n}^{ab\cdots c}, K_q^p, k_v^u$ that eliminates the variables $a, b, \cdots, c, l, m, \cdots, n, p, q, u, v$, and, moreover, k_v^u is the resolvent kernel of K_v^u .

A relative functional invariant of weight zero shall be termed an *absolute functional invariant*. In regard to absolute functional invariants we have the following theorem.†

If the Fredholm determinant $D[g_{a\beta}]$ of a quadratic functional form

$$(3.2) \quad g_{a\beta} y^a y^\beta + \int_a^b (g^a)^2 d\alpha$$

is not zero, the resolvent kernel $g^{a\beta}[g_{ab}]$ of $g_{a\beta}$,

$$(3.3) \quad g^{a\beta}[g_{ab}] = -\{D_{\beta^a}[g_{ab}]/D[g_{ab}]\},$$

is an absolute functional invariant of the quadratic form (3.2). The law of transformation of this functional invariant is

$$(3.4) \quad g^{a\beta}[\bar{g}_{ab}] = g^{a\beta}[g_{ab}] + g^{\gamma\beta}[g_{ab}] k_{\gamma}^a + g^{a\delta}[g_{ab}] k_{\delta}^{\beta} \\ + g^{\gamma\delta}[g_{ab}] k_{\gamma}^a k_{\delta}^{\beta} + k_{\beta}^a + k_a^{\beta} + \int_a^b k_{\sigma}^a k_{\sigma}^{\beta} d\sigma.$$

In order to establish a theorem which follows as a direct consequence

* This notion is due to A. D. Michal.

† Cf. A. D. Michal, *loc. cit.*

of the invariative property of the resolvent kernel $g^{\alpha\beta}[g_{ab}]$ of $g_{a\beta}$, let us consider the identity *

$$(3.5) \quad \begin{aligned} & D_{\beta_1\beta_2\ldots\beta_n}^{\alpha_1\alpha_2\ldots\alpha_n}[g_{ab}] + g_{a_1\sigma} D_{\beta_1\beta_2\ldots\beta_n}^{\sigma\alpha_2\ldots\alpha_n}[g_{ab}] \\ &= g_{a_1\beta_1} D_{\beta_2\beta_3\ldots\beta_n}^{\alpha_2\alpha_3\ldots\alpha_n}[g_{ab}] - g_{a_1\beta_2} D_{\beta_1\beta_3\ldots\beta_n}^{\alpha_2\alpha_3\ldots\alpha_n}[g_{ab}] + \cdots \\ & \quad + (-1)^{n-1} g_{a_1\beta_n} D_{\beta_1\beta_2\ldots\beta_{n-1}}^{\alpha_2\alpha_3\ldots\alpha_n}[g_{ab}], \end{aligned}$$

satisfied by the Fredholm minors. If we consider all the quantities α and β to be constant, with the exception of α_1 , we readily observe that (3.5) reduces to an integral equation of the form

$$f^{a_1} + g_{a_1\sigma} f^\sigma = \phi^{a_1},$$

where

$$(3.6) \quad \begin{cases} f^x = D_{\beta_1\beta_2\ldots\beta_n}^{\alpha_2\alpha_3\ldots\alpha_n}[g_{ab}], \\ \phi^x = g_{x\beta_1} D_{\beta_2\beta_3\ldots\beta_n}^{\alpha_2\alpha_3\ldots\alpha_n}[g_{ab}] - g_{x\beta_2} D_{\beta_1\beta_3\ldots\beta_n}^{\alpha_2\alpha_3\ldots\alpha_n}[g_{ab}] \\ \quad + \cdots + (-1)^{n-1} g_{x\beta_n} D_{\beta_1\beta_2\ldots\beta_{n-1}}^{\alpha_2\alpha_3\ldots\alpha_n}[g_{ab}]. \end{cases}$$

Hence, we have the unique continuous solution

$$f^{a_1} = \phi^{a_1} + \int_a^b g_{a_1\sigma} [g_{ab}] \phi^\sigma d\sigma,$$

which may be written in the form

$$(3.7) \quad D[g_{ab}]f^{a_1} = D[g_{ab}]\phi^{a_1} - D\sigma^{a_1}[g_{ab}]\phi^\sigma.$$

This follows from (3.3) and the fact that $D[g_{ab}] \neq 0$. Interpreting (3.7) according to (3.6) we have

$$\begin{aligned} D[g_{ab}] D_{\beta_1\beta_2\ldots\beta_n}^{\alpha_2\alpha_3\ldots\alpha_n}[g_{ab}] &= \{D[g_{ab}]g_{a_1\beta_1} - \int_a^b D\sigma^{a_1}[g_{ab}]g_{\sigma\beta_1}d\sigma\} \\ & \quad \times D_{\beta_2\beta_3\ldots\beta_n}^{\alpha_2\alpha_3\ldots\alpha_n}[g_{ab}] \\ & - \{D[g_{ab}]g_{a_1\beta_2} - \int_a^b D\sigma^{a_1}[g_{ab}]g_{\sigma\beta_2}d\sigma\} D_{\beta_1\beta_3\ldots\beta_n}^{\alpha_2\alpha_3\ldots\alpha_n}[g_{ab}] + \cdots \\ & + (-1)^{n-1} \{D[g_{ab}]g_{a_1\beta_n} - \int_a^b D\sigma^{a_1}[g_{ab}]g_{\sigma\beta_n}d\sigma\} D_{\beta_1\beta_2\ldots\beta_{n-1}}^{\alpha_2\alpha_3\ldots\alpha_n}[g_{ab}]. \end{aligned}$$

Simplifying each expression in brackets by the identity

$$D[g_{ab}]g_{a\beta} - \int_a^b D\sigma^a[g_{ab}]g_{\sigma\beta}d\sigma = D\beta^a[g_{ab}],$$

* Cf. R. D. D'Adhemar, *Leçons sur les principes de l'analyse*, Paris (1912), p. 237.

we obtain

$$(3.8) \quad D[g_{ab}] D_{\beta_1 \beta_2 \dots \beta_n}^{a_1 a_2 \dots a_n} [g_{ab}] = D_{\beta_1}^{a_1} [g_{ab}] D_{\beta_2 \beta_3 \dots \beta_n}^{a_2 a_3 \dots a_n} [g_{ab}] \\ - D_{\beta_2}^{a_1} [g_{ab}] D_{\beta_1 \beta_3 \dots \beta_n}^{a_2 a_3 \dots a_n} [g_{ab}] + \dots \\ + (-1)^{n-1} D_{\beta_n}^{a_1} [g_{ab}] D_{\beta_1 \beta_2 \dots \beta_{n-1}}^{a_2 a_3 \dots a_n} [g_{ab}].$$

In order to obtain (3.8) in the desired form, consider the case in which $n = 2$

$$(3.9) \quad D[g_{ab}] D_{\beta_1 \beta_2}^{a_1 a_2} [g_{ab}] = D_{\beta_1}^{a_1} [g_{ab}] D_{\beta_2}^{a_2} [g_{ab}] - D_{\beta_2}^{a_1} [g_{ab}] D_{\beta_1}^{a_2} [g_{ab}] \\ = \begin{vmatrix} D_{\beta_1}^{a_1} [g_{ab}], & D_{\beta_2}^{a_1} [g_{ab}] \\ D_{\beta_1}^{a_2} [g_{ab}], & D_{\beta_2}^{a_2} [g_{ab}] \end{vmatrix}.$$

Let us assume, then, that this form holds in general, i. e.

$$(3.10) \quad \{D[g_{ab}]\}^{n-1} D_{\beta_1 \beta_2 \dots \beta_n}^{a_1 a_2 \dots a_n} [g_{ab}] = \begin{vmatrix} D_{\beta_1}^{a_1} [g_{ab}], & D_{\beta_2}^{a_1} [g_{ab}], & \dots, & D_{\beta_n}^{a_1} [g_{ab}] \\ D_{\beta_1}^{a_2} [g_{ab}], & D_{\beta_2}^{a_2} [g_{ab}], & \dots, & D_{\beta_n}^{a_2} [g_{ab}] \\ \dots & \dots & \dots & \dots \\ D_{\beta_1}^{a_n} [g_{ab}], & D_{\beta_2}^{a_n} [g_{ab}], & \dots, & D_{\beta_n}^{a_n} [g_{ab}] \end{vmatrix}.$$

We shall proceed to prove by mathematical induction that the formula (3.10) is true. From (3.8) we have

$$D[g_{ab}] D_{\beta_1 \beta_2 \dots \beta_{n+1}}^{a_1 a_2 \dots a_{n+1}} [g_{ab}] = D_{\beta_1}^{a_1} [g_{ab}] D_{\beta_2 \beta_3 \dots \beta_{n+1}}^{a_2 a_3 \dots a_{n+1}} [g_{ab}] \\ - D_{\beta_2}^{a_1} [g_{ab}] D_{\beta_1 \beta_3 \dots \beta_{n+1}}^{a_2 a_3 \dots a_{n+1}} [g_{ab}] + \dots + (-1)^n D_{\beta_{n+1}}^{a_1} [g_{ab}] D_{\beta_1 \beta_2 \dots \beta_n}^{a_2 a_3 \dots a_{n+1}} [g_{ab}].$$

Multiplying both sides of the above equation by $\{D[g_{ab}]\}^{n-1}$ we readily observe, with the aid of (3.10) that we may obtain

$$\{D[g_{ab}]\}^n D_{\beta_1 \beta_2 \dots \beta_{n+1}}^{a_1 a_2 \dots a_{n+1}} [g_{ab}] = \begin{vmatrix} D_{\beta_1}^{a_1} [g_{ab}], & D_{\beta_2}^{a_1} [g_{ab}], & \dots, & D_{\beta_{n+1}}^{a_1} [g_{ab}] \\ D_{\beta_1}^{a_2} [g_{ab}], & D_{\beta_2}^{a_2} [g_{ab}], & \dots, & D_{\beta_{n+1}}^{a_2} [g_{ab}] \\ \dots & \dots & \dots & \dots \\ D_{\beta_1}^{a_{n+1}} [g_{ab}], & D_{\beta_2}^{a_{n+1}} [g_{ab}], & \dots, & D_{\beta_{n+1}}^{a_{n+1}} [g_{ab}] \end{vmatrix}.$$

Hence we see that the formula (3.10)* holds in general.

If we divide both sides of (3.10) by $\{D[g_{ab}]\}^n$ and take into consideration (3.3), we shall obtain

* This result is due to Hurwitz, *Bulletin of the American Mathematical Society*, Vol. 20 (1914), p. 408. For an indirect proof cf. Platrier, "Sur les mineurs de la fonction déterminante de Fredholm et sur les systèmes d'équations intégrales linéaires," *Journal de Mathématiques* (6e serie), tome 9 (1913), pp. 248-249.

$$(3.15) \quad g^{a_1\beta_1, a_2\beta_2}[\bar{g}_{ab}] = g^{\lambda_1\mu_1, \lambda_2\mu_2}[g_{ab}] \begin{pmatrix} a_1 \\ \lambda_1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \mu_1 \end{pmatrix} \begin{pmatrix} a_2 \\ \lambda_2 \end{pmatrix} \begin{pmatrix} \beta_2 \\ \mu_2 \end{pmatrix} \\ - \begin{vmatrix} 0 & \begin{pmatrix} \beta_1 \\ \mu_1 \end{pmatrix} \begin{pmatrix} \beta_2 \\ \mu_2 \end{pmatrix} \\ \begin{pmatrix} a_1 \\ \lambda_1 \end{pmatrix} & t_{\beta_1}^{a_1} & t_{\beta_2}^{a_1} \\ \begin{pmatrix} a_2 \\ \lambda_2 \end{pmatrix} & t_{\beta_1}^{a_2} & t_{\beta_2}^{a_2} \end{vmatrix} g^{\lambda_1\mu_1}[g_{ab}] + \begin{vmatrix} t_{\beta_1}^{a_1} & t_{\beta_2}^{a_1} \\ t_{\beta_1}^{a_2} & t_{\beta_2}^{a_2} \end{vmatrix}.$$

Let us now consider the third resolvent, i. e.,

$$(3.16) \quad g^{a_1\beta_1, a_2\beta_2, a_3\beta_3}[\bar{g}_{ab}] = \begin{vmatrix} g^{a_1\beta_1}[\bar{g}_{ab}], & g^{a_1\beta_2}[\bar{g}_{ab}], & g^{a_1\beta_3}[\bar{g}_{ab}] \\ g^{a_2\beta_1}[\bar{g}_{ab}], & g^{a_2\beta_2}[\bar{g}_{ab}], & g^{a_2\beta_3}[\bar{g}_{ab}] \\ g^{a_3\beta_1}[\bar{g}_{ab}], & g^{a_3\beta_2}[\bar{g}_{ab}], & g^{a_3\beta_3}[\bar{g}_{ab}] \end{vmatrix}.$$

In precisely the same manner as for (3.14), we substitute (3.13) for the resolvent kernels in (3.16). Expanding the determinant and collecting like terms, we obtain

$$(3.17) \quad g^{a_1\beta_1, a_2\beta_2, a_3\beta_3}[\bar{g}_{ab}] \\ = g^{\lambda_1\mu_1, \lambda_2\mu_2, \lambda_3\mu_3}[g_{ab}] \begin{pmatrix} a_1 \\ \lambda_1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \mu_1 \end{pmatrix} \begin{pmatrix} a_2 \\ \lambda_2 \end{pmatrix} \begin{pmatrix} \beta_2 \\ \mu_2 \end{pmatrix} \begin{pmatrix} a_3 \\ \lambda_3 \end{pmatrix} \begin{pmatrix} \beta_3 \\ \mu_3 \end{pmatrix} \\ + g^{\lambda_1\mu_1, \lambda_2\mu_2}[g_{ab}] \\ \left\{ \begin{pmatrix} a_1 \\ \lambda_1 \end{pmatrix} \begin{pmatrix} a_2 \\ \lambda_2 \end{pmatrix} \left[\begin{pmatrix} \beta_1 \\ \mu_1 \end{pmatrix} \begin{pmatrix} \beta_2 \\ \mu_2 \end{pmatrix} t_{\beta_3}^{a_3} - \begin{pmatrix} \beta_1 \\ \mu_1 \end{pmatrix} \begin{pmatrix} \beta_3 \\ \mu_3 \end{pmatrix} t_{\beta_2}^{a_3} + \begin{pmatrix} \beta_2 \\ \mu_2 \end{pmatrix} \begin{pmatrix} \beta_3 \\ \mu_3 \end{pmatrix} t_{\beta_1}^{a_3} \right] \right. \\ + \begin{pmatrix} a_2 \\ \lambda_2 \end{pmatrix} \begin{pmatrix} a_3 \\ \lambda_3 \end{pmatrix} \left[\begin{pmatrix} \beta_1 \\ \mu_1 \end{pmatrix} \begin{pmatrix} \beta_2 \\ \mu_2 \end{pmatrix} t_{\beta_3}^{a_1} - \begin{pmatrix} \beta_1 \\ \mu_1 \end{pmatrix} \begin{pmatrix} \beta_3 \\ \mu_3 \end{pmatrix} t_{\beta_2}^{a_1} + \begin{pmatrix} \beta_2 \\ \mu_2 \end{pmatrix} \begin{pmatrix} \beta_3 \\ \mu_3 \end{pmatrix} t_{\beta_1}^{a_1} \right] \\ \left. + \begin{pmatrix} a_3 \\ \lambda_3 \end{pmatrix} \begin{pmatrix} a_1 \\ \lambda_1 \end{pmatrix} \left[\begin{pmatrix} \beta_1 \\ \mu_1 \end{pmatrix} \begin{pmatrix} \beta_2 \\ \mu_2 \end{pmatrix} t_{\beta_3}^{a_2} - \begin{pmatrix} \beta_1 \\ \mu_1 \end{pmatrix} \begin{pmatrix} \beta_3 \\ \mu_3 \end{pmatrix} t_{\beta_2}^{a_2} + \begin{pmatrix} \beta_2 \\ \mu_2 \end{pmatrix} \begin{pmatrix} \beta_3 \\ \mu_3 \end{pmatrix} t_{\beta_1}^{a_2} \right] \right\} \\ + g^{\lambda_1\mu_1}[g_{ab}] \\ \left(\begin{pmatrix} a_1 \\ \lambda_1 \end{pmatrix} \left\{ \begin{pmatrix} \beta_1 \\ \mu_1 \end{pmatrix} \left[t_{\beta_2}^{a_2} t_{\beta_3}^{a_3} - t_{\beta_3}^{a_2} t_{\beta_2}^{a_3} \right] - \begin{pmatrix} \beta_2 \\ \mu_2 \end{pmatrix} \left[t_{\beta_1}^{a_2} t_{\beta_3}^{a_3} - t_{\beta_3}^{a_2} t_{\beta_1}^{a_3} \right] \right. \right. \\ + \begin{pmatrix} \beta_3 \\ \mu_3 \end{pmatrix} \left[t_{\beta_1}^{a_2} t_{\beta_2}^{a_3} - t_{\beta_2}^{a_2} t_{\beta_1}^{a_3} \right] \left. \right\} - \begin{pmatrix} a_2 \\ \lambda_2 \end{pmatrix} \left\{ \begin{pmatrix} \beta_1 \\ \mu_1 \end{pmatrix} \left[t_{\beta_2}^{a_1} t_{\beta_3}^{a_3} - t_{\beta_3}^{a_1} t_{\beta_2}^{a_3} \right] \right. \\ - \begin{pmatrix} \beta_2 \\ \mu_2 \end{pmatrix} \left[t_{\beta_1}^{a_1} t_{\beta_3}^{a_3} - t_{\beta_3}^{a_1} t_{\beta_1}^{a_3} \right] + \begin{pmatrix} \beta_3 \\ \mu_3 \end{pmatrix} \left[t_{\beta_1}^{a_1} t_{\beta_2}^{a_3} - t_{\beta_2}^{a_1} t_{\beta_1}^{a_3} \right] \left. \right\} \\ \left. + \begin{pmatrix} a_3 \\ \lambda_3 \end{pmatrix} \left\{ \begin{pmatrix} \beta_1 \\ \mu_1 \end{pmatrix} \left[t_{\beta_2}^{a_1} t_{\beta_3}^{a_2} - t_{\beta_3}^{a_1} t_{\beta_2}^{a_2} \right] - \begin{pmatrix} \beta_2 \\ \mu_2 \end{pmatrix} \left[t_{\beta_1}^{a_1} t_{\beta_3}^{a_2} - t_{\beta_3}^{a_1} t_{\beta_1}^{a_2} \right] \right. \right. \right.$$

$$\begin{aligned}
& + \left(\beta_3 \right)_{\mu_1} [t_{\beta_1}^{a_1} t_{\beta_2}^{a_2} - t_{\beta_2}^{a_1} t_{\beta_1}^{a_2}] \} \\
& + [t_{\beta_1}^{a_1} t_{\beta_2}^{a_2} t_{\beta_3}^{a_3} + t_{\beta_2}^{a_1} t_{\beta_3}^{a_2} t_{\beta_1}^{a_3} + t_{\beta_3}^{a_1} t_{\beta_1}^{a_2} t_{\beta_2}^{a_3} \\
& - t_{\beta_3}^{a_1} t_{\beta_2}^{a_2} t_{\beta_1}^{a_3} - t_{\beta_2}^{a_1} t_{\beta_1}^{a_2} t_{\beta_3}^{a_3} - t_{\beta_1}^{a_1} t_{\beta_3}^{a_2} t_{\beta_2}^{a_3}].
\end{aligned}$$

In the same manner as for the second resolvent, we may write (3.17) in the following form,

$$\begin{aligned}
(3.18) \quad g^{a_1 \beta_1, a_2 \beta_2, a_3 \beta_3} [\bar{g}_{ab}] &= g^{\lambda_1 \mu_1, \lambda_2 \mu_2, \lambda_3 \mu_3} [g_{ab}] \begin{pmatrix} a_1 \\ \lambda_1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \mu_1 \end{pmatrix} \begin{pmatrix} a_2 \\ \lambda_2 \end{pmatrix} \begin{pmatrix} \beta_2 \\ \mu_2 \end{pmatrix} \begin{pmatrix} a_3 \\ \lambda_3 \end{pmatrix} \begin{pmatrix} \beta_3 \\ \mu_3 \end{pmatrix} \\
& + [1/(2!)^2] \begin{vmatrix} 0 & 0 & \begin{pmatrix} \beta_1 \\ \mu_1 \end{pmatrix} & \begin{pmatrix} \beta_2 \\ \mu_1 \end{pmatrix} & \begin{pmatrix} \beta_3 \\ \mu_1 \end{pmatrix} \\ 0 & 0 & \begin{pmatrix} \beta_1 \\ \mu_2 \end{pmatrix} & \begin{pmatrix} \beta_2 \\ \mu_2 \end{pmatrix} & \begin{pmatrix} \beta_3 \\ \mu_2 \end{pmatrix} \\ \begin{pmatrix} a_1 \\ \lambda_1 \end{pmatrix} & \begin{pmatrix} a_1 \\ \lambda_2 \end{pmatrix} & t_{\beta_1}^{a_1} & t_{\beta_2}^{a_1} & t_{\beta_3}^{a_1} \\ \begin{pmatrix} a_2 \\ \lambda_1 \end{pmatrix} & \begin{pmatrix} a_2 \\ \lambda_2 \end{pmatrix} & t_{\beta_1}^{a_2} & t_{\beta_2}^{a_2} & t_{\beta_3}^{a_2} \\ \begin{pmatrix} a_3 \\ \lambda_1 \end{pmatrix} & \begin{pmatrix} a_3 \\ \lambda_2 \end{pmatrix} & t_{\beta_1}^{a_3} & t_{\beta_2}^{a_3} & t_{\beta_3}^{a_3} \end{vmatrix} g^{\lambda_1 \mu_1, \lambda_2 \mu_2} [g_{ab}] \\
- \begin{vmatrix} 0 & \begin{pmatrix} \beta_1 \\ \mu_1 \end{pmatrix} & \begin{pmatrix} \beta_2 \\ \mu_1 \end{pmatrix} & \begin{pmatrix} \beta_3 \\ \mu_1 \end{pmatrix} \\ \begin{pmatrix} a_1 \\ \lambda_1 \end{pmatrix} & t_{\beta_1}^{a_1} & t_{\beta_2}^{a_1} & t_{\beta_3}^{a_1} \\ \begin{pmatrix} a_2 \\ \lambda_1 \end{pmatrix} & t_{\beta_1}^{a_2} & t_{\beta_2}^{a_2} & t_{\beta_3}^{a_2} \\ \begin{pmatrix} a_3 \\ \lambda_1 \end{pmatrix} & t_{\beta_1}^{a_3} & t_{\beta_2}^{a_3} & t_{\beta_3}^{a_3} \end{vmatrix} g^{\lambda_1 \mu_1} [g_{ab}] + \begin{vmatrix} t_{\beta_1}^{a_1} & t_{\beta_2}^{a_1} & t_{\beta_3}^{a_1} \\ t_{\beta_1}^{a_2} & t_{\beta_2}^{a_2} & t_{\beta_3}^{a_2} \\ t_{\beta_1}^{a_3} & t_{\beta_2}^{a_3} & t_{\beta_3}^{a_3} \end{vmatrix}
\end{aligned}$$

As for the two preceding resolvents, we can find a similar expression for the fourth resolvent. It is highly probable that a similar expression holds good for the n th resolvent. A proof of such a relation however appears to be difficult.

THEOREM II. *The first i resolvents $g^{a_1 \beta_1} [g_{ab}], \dots, g^{a_i \beta_i}, \dots, g^{a_i \beta_i} [g_{ab}]$, ($i = 1, 2, 3, 4$) form an absolute invariant class of i functionals.*

If we inquire as to the meaning of

$$(3.19) \quad t_{\beta}^a = k_{\beta}^a + k_a^{\beta} + \int_a^b k_{\sigma}^a k_{\sigma}^{\beta} d\sigma = 0$$

we find that this condition gives us the orthogonal group of Fredholm transformations. The characteristic property of this group is ordinarily given as

$$K_{\beta}^{\alpha} + K_{\alpha}^{\beta} + \int_a^b K_{\alpha}^{\sigma} K_{\beta}^{\sigma} d\sigma = 0$$

which immediately yields (3.19), by noting the fact that K_{α}^{β} is the resolvent kernel of K_{β}^{α} and by making use of a theorem of Evans.*

Under the orthogonal Fredholm group we note that (3.13) reduces to

$$(3.20) \quad g^{a\beta}[\bar{g}_{ab}] = g^{\lambda\mu}[g_{ab}] \begin{pmatrix} a \\ \lambda \end{pmatrix} \begin{pmatrix} \beta \\ \mu \end{pmatrix}.$$

Substituting (3.20) in (3.11), we obtain

$$(3.21) \quad g^{a_1 \beta_1, a_2 \beta_2, \dots, a_n \beta_n} [\bar{g}_{ab}] \\ = g^{\lambda_1 \mu_1, \lambda_2 \mu_2, \dots, \lambda_n \mu_n} [g_{ab}] \begin{pmatrix} a_1 \\ \lambda_1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \mu_1 \end{pmatrix} \begin{pmatrix} a_2 \\ \lambda_2 \end{pmatrix} \begin{pmatrix} \beta_2 \\ \mu_2 \end{pmatrix} \dots \begin{pmatrix} a_n \\ \lambda_n \end{pmatrix} \begin{pmatrix} \beta_n \\ \mu_n \end{pmatrix}$$

which gives us the general law of transformation of the n th resolvent under the orthogonal Fredholm group of transformations.

While it may seem that the orthogonality property imposes a great restriction on the Fredholm group of transformations, we shall show that this group is exactly the same as the group of Fredholm transformations which leaves the general Gramian invariant.

Let us consider, then, that we have a Gramian which is left invariant under a Fredholm transformation. That is to say

$$(3.22) \quad \begin{vmatrix} (\bar{y}_1 \bar{y}_1), & (\bar{y}_1 \bar{y}_2), & \cdots, & (\bar{y}_1 \bar{y}_n) \\ (\bar{y}_2 \bar{y}_1), & (\bar{y}_2 \bar{y}_2), & \cdots, & (\bar{y}_2 \bar{y}_n) \\ \vdots & \vdots & \ddots & \vdots \\ (\bar{y}_n \bar{y}_1), & (\bar{y}_n \bar{y}_2), & \cdots, & (\bar{y}_n \bar{y}_n) \end{vmatrix} = \begin{vmatrix} (y_1 y_1), & (y_1 y_2), & \cdots, & (y_1 y_n) \\ (y_2 y_1), & (y_2 y_2), & \cdots, & (y_2 y_n) \\ \vdots & \vdots & \ddots & \vdots \\ (y_n y_1), & (y_n y_2), & \cdots, & (y_n y_n) \end{vmatrix}$$

where

$$(y_i y_k) = \int_a^b y_i^\sigma y_k^\sigma d\sigma, \quad (i, k = 1, 2, \dots, n).$$

Let us expand (3.22) in order to have it in the following form

$$(3.23) \quad (\bar{y}_1 \bar{y}_1) \begin{vmatrix} (\bar{y}_2 \bar{y}_2), \dots, (\bar{y}_2 \bar{y}_n) \\ \vdots \\ (\bar{y}_n \bar{y}_2), \dots, (\bar{y}_n \bar{y}_n) \end{vmatrix} + Q_{\alpha\beta} [\bar{y}_2, \bar{y}_3, \dots, \bar{y}_n] \bar{y}_1^\alpha \bar{y}_1^\beta$$

* Cf. G. C. Evans, "Functionals and Their Applications," *Cambridge Colloquium Lectures*, New York (1918), pp. 122.

By the fundamental lemma of the calculus of variations, we see that (3.25) implies

$$(3.27) \quad \Omega[\bar{y}_2, \dots, \bar{y}_n, y_2, \dots, y_n] \bar{y}_1^\alpha + \Lambda_\beta^\alpha[\bar{y}_2, \dots, \bar{y}_n, y_2, \dots, y_n, K_\mu^\lambda] \bar{y}_1^\beta = 0.$$

In order, however, that this relation be true, we must have *

$$\Omega[\bar{y}_2, \dots, \bar{y}_n, y_2, \dots, y_n] = 0, \quad \Lambda_\beta^\alpha[\bar{y}_2, \dots, \bar{y}_n, y_2, \dots, y_n, K_\mu^\lambda] = 0.$$

Considering (3.26), we see that $\Omega = 0$ implies the invariance of $(n-1)$ rowed minor of our Gramian. Hence, a necessary condition that a Gramian of the n th order be an absolute invariant under the Fredholm group of transformations, is that the Gramian of the $(n-1)$ degree be absolutely invariant.

Repeating the above operation $(n-1)$ times, we see that the proof of our problem depends on the invariance of a Gramian of the first order. Consider, then, the Gramian of order one

$$(3.28) \quad \int_a^b y_i^\alpha y_k^\alpha d\alpha = \int_a^b \bar{y}_i^\alpha \bar{y}_k^\alpha d\alpha.$$

Under the Fredholm group of transformations, we obtain the condition

$$\int_a^b \int_a^b [K_\beta^\alpha + K_\alpha^\beta + \int_a^b K_\alpha^\sigma K_\beta^\sigma d\sigma] \bar{y}_i^\alpha \bar{y}_k^\beta d\alpha d\beta = 0,$$

in order that (3.28) be true. This, however, implies

$$(3.29) \quad K_\beta^\alpha + K_\alpha^\beta + \int_a^b K_\alpha^\sigma K_\beta^\sigma d\sigma = 0,$$

as we have seen above. Equation (3.29) is the characteristic equation of the orthogonal Fredholm group of transformations and hence a necessary condition that a Gramian of the n th degree be an absolute invariant is that the Fredholm group of transformations be orthogonal.

Interpreting (3.28) as a general element of a Gramian of the n th degree, we see that the above necessary condition is also sufficient; since when each entry in a determinant is an absolute invariant, it follows that the determinant is an absolute invariant.

THEOREM III. *A necessary and sufficient condition that a Gramian be an absolute invariant under the Fredholm group of transformations is that the group of transformations be orthogonal.*

* Cf. G. Kowalewski, "Über Funktionenräume," *Sitzb. d. mathem. Naturw. Kl.*, CXX Bd., Abt. IIa, Novem. 1911, pp. 1455.

4. *The Fredholm group with a parameter.* Let us consider a Fredholm transformation with an arbitrary parameter λ ,

$$(4.1) \quad y^a = \bar{y}^a + \lambda K_{\sigma}^a \bar{y}^{\sigma},$$

such that λ is not a characteristic value of the kernel K_b^a . The totality of such Fredholm transformations form a group. In fact, let us consider two arbitrary transformations of the type (4.1);

$$T_1 : \bar{y}^a = y^a + \lambda A_{\sigma}^a y^{\sigma}$$

$$T_2 : \bar{y}^a = \bar{y}^a + \mu B_{\tau}^a \bar{y}^{\tau}$$

where λ and μ are not characteristic values for their respective kernels.

By taking the product $T_1 T_2$ of the transformations T_1 and T_2 , we obtain

$$\bar{y}^a = y^a + K_{\sigma}^a y^{\sigma},$$

where

$$K_{\sigma}^a = \lambda A_{\sigma}^a + \mu B_{\sigma}^a + \lambda \mu B_{\tau}^a A_{\sigma}^{\tau}.$$

By a theorem due to Jonesco,* we see that the kernel K_b^a has for a Fredholm determinant, corresponding to the value 1 of the parameter,

$$D_K(1) = D_A(\lambda) \cdot D_B(\mu).$$

Since by hypothesis $D_A(\lambda) \neq 0$ and $D_B(\mu) \neq 0$, we readily observe that $D_K(1)$ is different from zero and hence that the transformations (4.1) satisfy the closure property.

Since $D_A(\lambda) \neq 0$, we have the unique, continuous inverse †

$$y^a = \bar{y}^a + \lambda \mathcal{A}_{\sigma}^a \bar{y}^{\sigma},$$

where

$$\mathcal{A}_{\sigma}^a = - [D_{\sigma}^a(\lambda) / D_A(\lambda)].$$

Obviously the kernel, zero, gives us the identical transformation. We may state then the

THEOREM IV. *The Fredholm transformations with a parameter*

$$y^a = \bar{y}^a + \lambda K_{\sigma}^a \bar{y}^{\sigma},$$

where λ is not a characteristic value of the kernel K_b^a constitute a group.

* Cf. Jonesco, "Sur la fonction caractéristique et le noyau résolvant d'un noyau donné," *Bulletin des Sciences Mathématique*, 2e série, tome 50 (Juillet, 1926).

† Cf. Goursat, *Cours d'Analyse*, Paris (1927), Vol. III, pp. 373.

5. *Cogredient and contragredient functional transformations.* We shall say that the functions u^a, v^a, \dots transform cogrediently to the function y^i if

$$(5.1) \quad \begin{aligned} \bar{u}^a &= u^a + K_{\sigma}{}^a u^{\sigma} \\ \bar{v}^a &= v^a + K_{\sigma}{}^a v^{\sigma} \\ &\text{etc.} \end{aligned}$$

when

$$\bar{y}^i = y^i + K_a{}^i y^a$$

where the kernel $K_b{}^a$ is the same in both transformations.

In addition, we shall say that the functions ξ_a, η_a, \dots transform contragrediently to the function y^i , if

$$(5.2) \quad \begin{aligned} \bar{\xi}_a &= \xi_a + \int_a^b k_{\sigma}{}^a \xi_{\sigma} d\sigma, \\ \bar{\eta}_a &= \eta_a + \int_a^b k_{\sigma}{}^a \eta_{\sigma} d\sigma \\ &\text{etc.} \end{aligned}$$

where $k_b{}^a$ is the resolvent kernel of the kernel $K_b{}^a$.

With these conceptions clear in mind then, let us investigate the induced transformation in the function $h^{a\beta}$ when the quadratic functional form

$$h^{a\beta} \xi_a \xi_{\beta} + \int_a^b (\xi_a)^2 d\alpha, \quad (h^{a\beta} = h^{\beta a})$$

is an absolute invariant under the contragredient transformations (5.2). This problem is analogous to that of equation (2.9) and, as a result, we obtain

$$(5.3) \quad \begin{aligned} h^{a\beta} &= h^{a\beta} + h^{\gamma\beta} k_{\gamma}{}^a + h^{a\delta} k_{\delta}{}^{\beta} + h^{\gamma\delta} k_{\gamma}{}^a k_{\delta}{}^{\beta} \\ &\quad + k_{\beta}{}^a + k_a{}^{\beta} + \int_a^b k_{\sigma}{}^a k_{\sigma}{}^{\beta} d\sigma, \end{aligned}$$

which is precisely of the form (3.4), the law of transformation of the kernel $g^{a\beta}[g_{ab}]$ when the quadratic functional form (1.1) is absolutely invariant.

Thus we see that the resolvent kernel itself may be considered as a coefficient of a quadratic functional form

$$g^{a\beta}[g_{ab}] \xi_a \xi_{\beta} + \int_a^b (\xi_a)^2 d\alpha.$$

Let the functional $Q[\xi, \eta]$ be defined by

$$Q[\xi, \eta] = g^{a\beta}[g_{ab}] \xi_a \eta_{\beta} + \int_a^b \xi_a \eta_a d\alpha.$$

On the basis of the above result it is clear that the determinant

$$(5.4) \quad \Delta = \begin{vmatrix} Q[\xi, \xi], & Q[\xi, \eta] \\ Q[\eta, \xi], & Q[\eta, \eta] \end{vmatrix}$$

is an absolute quartic functional form under the contragredient transformations (5.2). The determinant Δ when expanded can be written in the form

$$(5.5) \quad g^{a_1 \beta_1, a_2 \beta_2}[g_{ab}] \xi_{a_1} \xi_{\beta_1} \eta_{a_2} \eta_{\beta_2} \\ + g^{a_1 \beta_1}[g_{ab}] \begin{vmatrix} \xi_{a_1} \xi_{\sigma_1} \\ \eta_{a_1} \eta_{\sigma_1} \end{vmatrix} \cdot \begin{vmatrix} \xi_{\beta_1} \xi_{\sigma_1} \\ \eta_{\beta_1} \eta_{\sigma_1} \end{vmatrix} + \xi_{\sigma_2} \eta_{\sigma_1} \begin{vmatrix} \xi_{\sigma_2} \xi_{\sigma_1} \\ \eta_{\sigma_2} \eta_{\sigma_1} \end{vmatrix} \cdot \\ \text{(integration on } \sigma_1 \text{ and } \sigma_2)$$

This can be verified most easily by substituting for $g^{a_1 \beta_1, a_2 \beta_2}[g_{ab}]$ its expression in terms of $g^{a\beta}[g_{ab}]$ as given by formula (3.11).

These results then imply the following interesting theorem:

THEOREM V. *The second resolvent $g^{a_1 \beta_1, a_2 \beta_2}[g_{ab}]$ and the resolvent kernel $g^{a\beta}[g_{ab}]$ of an absolute quadratic functional form (1.1) may be taken as coefficients of the absolute quartic functional form (5.5).*

THE OHIO STATE UNIVERSITY.

On the Contact of a Quartic Surface with an Analytic Surface.

BY ERNEST P. LANE.

1. *Introduction.* An algebraic surface A is said to have contact of order k with an analytic surface S at a point P on S in case every curve on S through P has $k + 1$ consecutive points on A at P . If A and S are in ordinary space, and if S is represented in the neighborhood of P by a power series expansion of one non-homogeneous projective coordinate z of a point on the surface in terms of the other two coördinates x, y , with respect to a local coordinate system, while A is represented by an algebraic equation of degree h in x, y, z , then this equation is satisfied identically in x, y as far as the terms of degrees k when the series for z is substituted therein.

It is $(k + 1)(k + 2)/2$ conditions for an algebraic surface to have contact of order k at a point of an analytic surface. Since an algebraic surface of order h depends on $h(h^2 + 6h + 11)/6$ parameters there is a limitation on the order of contact possible for a given algebraic surface. For instance, a quadric surface, being of order two, depends on nine parameters. Since contact of the second order imposes six conditions, there are ∞^3 quadrics having contact of the second order with S at P . These quadrics play a prominent part in the projective differential geometry of a surface. But, since contact of the third order imposes ten conditions, there is no non-composite quadric having contact of the third order with S at P , if S is not itself a quadric.

A cubic surface depends on nineteen parameters, so that there are ∞^4 cubic surfaces having contact of the fourth order with S at P . There is no non-composite cubic surface having contact of the fifth order with S at P , if S is not itself a cubic. A quartic surface depends on thirty-four parameters, so that there are ∞^6 quartic surfaces having contact of the sixth order with S at P . There is no non-composite quartic surface having contact of the seventh order with S at P , if S is not itself a quartic.

It is the purpose of this paper to summarize briefly the most interesting results that have been obtained concerning cubic surfaces having contact of various orders at a point of an analytic surface, and then to initiate an investigation of the contact of a general quartic surface with an analytic surface. The cases in which the quartics have contact of the fourth, fifth, and sixth

orders receive particular attention. In each case the possibilities as to composite quartic surfaces are considered, and emphasis is laid on the curve of intersection of a non-composite quartic and the tangent plane at the point of contact.

2. *Analytic Basis.* If the four projective homogeneous coordinates $x^{(1)}, \dots, x^{(4)}$ of a point P on a non-degenerate non-ruled surface S in ordinary space are given as single valued analytic functions of two independent variables u, v , and if the parametric net on S is the asymptotic net, then the functions x when multiplied by a suitably chosen common factor, are solutions of a completely integrable system of differential equations in Fubini's canonical form,

$$(1) \quad x_{uu} = px + \theta_u x_u + \beta x_v, \quad x_{vv} = qx + \gamma x_u + \theta_v x_v, \quad (\theta = \log \beta \gamma).$$

The coordinates of a point Q on S near P can be represented by Taylor's formula as power series in the increments $\Delta u, \Delta v$ corresponding to displacement on S from P to Q . Then by means of (1) and the equations obtained from (1) by differentiation it is possible to express each of these series in the form $x_1 x + x_2 x_u + x_3 x_v + x_4 x_{uv}$, where x_1, \dots, x_4 are power series in $\Delta u, \Delta v$ which represent the local coordinates of Q referred to the covariant tetrahedron x, x_u, x_v, x_{uv} with suitably chosen unit point. It is easy to calculate power series in $\Delta u, \Delta v$ for the non-homogeneous coordinates x, y, z of Q defined by placing

$$(2) \quad x = x_2/x_1, \quad y = x_3/x_1, \quad z = x_4/x_1.$$

From these series it is possible to compute an expansion for z as a power series in x, y . To terms of the sixth degree the series thus obtained can be written * in the form

$$(3) \quad z = T + U + V + W + \dots,$$

where

$$(4) \quad \begin{aligned} T &= xy - (\beta x^3 + \gamma y^3)/3, \\ 12U &= \beta \phi x^4 - 4\beta \psi x^2 y - 6\theta_{uv} x^2 y^2 - 4\gamma \phi x y^3 + \gamma \psi y^4, \\ 60V &= \beta c_0 x^5 + 5\beta c_1 x^4 y + 10\beta c_2 x^3 y^2 + 10\gamma c_3 x^2 y^3 + 5\gamma c_4 x y^4 + \gamma c_5 y^5, \\ W &= \beta d_0 x^6 + \beta d_1 x^5 y + \beta d_2 x^4 y^2 + d_3 x^3 y^3 + \gamma d_4 x^2 y^4 + \gamma d_5 x y^5 + \gamma d_6 y^6, \end{aligned}$$

* Lane, "Power Series Expansions in the Neighborhood of a Point on a Surface," *Proceedings of the National Academy of Sciences*, Vol. 13 (1927), p. 803.

and

$$\begin{aligned}
 \phi &= (\log \beta \gamma^2)_u, & \psi &= (\log \beta^2 \gamma)_v, \\
 (5) \quad c_0 &= \phi_u - \theta_u \phi - \phi^2 + 8\beta\psi - 4p, & c_5 &= \psi_v - \theta_v \psi - \psi^2 + 8\gamma\phi - 4q, \\
 c_1 &= \phi\psi - \psi_u + 7\theta_{uv} + 4\beta\gamma, & c_4 &= \phi\psi - \phi_v + 7\theta_{uv} + 4\beta\gamma, \\
 c_2 &= 3\gamma\phi - \psi^2 + \theta_v \psi - \psi_v + 2q, & c_3 &= 3\beta\psi - \phi^2 + \theta_u \phi - \phi_u + 2p,
 \end{aligned}$$

the coefficients d_0, \dots, d_6 having never been calculated in terms of the coefficients of (1). The series (3) is the basis of the investigations of the present paper.

3. *Cubic Surfaces.* If we write the most general equation of the third degree in x, y, z and demand that this equation be satisfied by the expression for z in (3) identically in x, y as far as the terms of the fourth degree, the result can be written in the form

$$(6) \quad A[z - T + \theta_{uv}z^2/2 + (z - xy)(\phi x + \psi y)/4 + 5z(\beta\psi x^2 + \gamma\phi y^2)/12] + Bz^2 + Cxz^2 + Dyz^2 + Ez(z - xy) = 0,$$

the coefficients, A, B, C, D, E , being arbitrary. Every one of the cubic surfaces represented by (6) has contact of the fourth order with S at P . There are ∞^3 of these surfaces which are composite, one component of each being the tangent plane $z = 0$ and the other being a quadric surface having contact of the second order with S at P ; for these surfaces, $A = 0$. *All the other ∞^1 cubic surfaces represented by (6) cut the tangent plane in the same cubic curve whose equations are*

$$(7) \quad z = xy - (\beta x^3 + \gamma y^3)/3 + xy(\phi x + \psi y)/4 = 0.$$

This is a covariant cubic curve having a node at P , with the asymptotic tangents for double-point tangents, and having three inflexions which lie on a straight line whose equations in homogeneous coordinates are found by (2) to be

$$(8) \quad x_4 = 4x_1 + \phi x_2 + \psi x_3 = 0.$$

This is the second canonical edge of Green. One of the points of inflexion lies on each of the tangents of Darboux,*

$$(9) \quad x_4 = \beta x_2^3 + \gamma x_3^3 = 0.$$

If we demand that (6) be satisfied by (3) identically in x, y as far as

* B. Segre, "La cubique indicatrice de l'élément linéaire projective d'une surface," *Comptes Rendus*, Vol. 184 (1927), p. 729.

the terms of the fifth degree without restriction on S , we obtain the tangent plane counted three times as the only cubic surface having contact of the fifth order with S at P . But if we suppose $A \neq 0$, then S is restricted to be itself a cubic, and necessary and sufficient conditions therefor are found to be

$$\begin{aligned}
 (10) \quad & (\log \beta/\gamma)_{uv} = 0, \\
 & 4p = \phi_u = \theta_u \phi + \phi^2/4 - \beta\psi/3, \\
 & 4q = \psi_v - \theta_v \psi + \psi^2/4 - \gamma\phi/3.
 \end{aligned}$$

By a suitable choice of parameters* it is possible to make $\beta = \gamma$ and consequently to simplify these conditions.

4. *Quartic surfaces with contact of the fourth order.* If we write the most general equation of the fourth degree in x, y, z and demand that this equation be satisfied by the expression for z in (3) identically in x, y as far as the terms of the fourth degree, the result can be written in the form

$$\begin{aligned}
 (11) \quad & (gx + fy)(z - T) + h(z - T - U) + c(z^2 - x^2y^2) + Cz^4 \\
 & + Ex^3z + Gy^3z + Hxz^3 + Iyz^3 + Kx^2z^2 + Ly^2z^2 + Mxyz^2 \\
 & + Nxy^2z + Qx^2yz + (z - xy)(E'x^2 + J'xy + G'y^2) \\
 & + C'z^3 + H'xz^2 + I'yz^2 = 0,
 \end{aligned}$$

the twenty coefficients g, f, h, \dots, H', I' being arbitrary. Every one of the ∞^{19} quartic surfaces represented by (11) has contact of the fourth order with S at P . There are ∞^{13} of these quartics which are composite, one component of each being the tangent plane $z = 0$ and the other being a cubic surface having contact of the second order with S at P . Any one of the ∞^6 other quartic surfaces represented by (11) intersects the tangent plane in a quartic curve whose equations are

$$(12) \quad z = (gx + fy)T + h(T + U) + xy[E'x^2 + (c + J')xy + G'y^2] = 0.$$

There are only ∞^5 of these curves, instead of the ∞^6 that might have been expected. Each of these curves for which $h = 0$ has a node at P , the double-point tangents being the asymptotic tangents $z = xy = 0$. A curve (12) for which $h = 0$, but not both of f, g are zero, has a triple-point at P , the triple-point tangents being the asymptotic tangents and the line $z = gx + fy = 0$. It is clear that if either of f and g is zero while the other is not, two of the

* Lane, "The Contact of a Cubic Surface with an Analytic Surface," *Transactions of the American Mathematical Society*, Vol. 29 (1927), p. 471.

triple-point tangents coincide in an asymptotic tangent. And if $f=g=h=0$, the curve is composite, the components being the asymptotic tangents and the two straight lines

$$(13) \quad z = E'x^2 + (c + J')xy + G'y^2 = 0.$$

There is another case of interest in which the curve (12) is composite, namely, the case in which this curve consists of two proper conics, one being tangent to each of the asymptotic tangents at P . Writing the equations of two such conics, multiplying them together, and comparing the result with (12), we easily find that *conditions necessary and sufficient that (12) with $h=1$ represent two proper conics are*

$$(14) \quad \begin{aligned} 12E' &= 4\beta f + 3\phi g + 3\beta\psi - 3\phi^2/4, \\ 12G' &= 3\phi f + 4\gamma g + 3\gamma\phi - 3\psi^2/4, \\ 12(c + J') &= 12fg - 3\phi f - 3\psi g + 3\phi\psi/2 \\ &\quad + 6\theta_{uv} + 4\beta\gamma/3. \end{aligned}$$

If every tangent plane of a quartic surface cuts the surface in two conics, the surface is a Roman surface of Steiner. Since such a surface depends on fifteen parameters, and since contact of the fourth order imposes fifteen conditions, it is not surprising that there should be a unique Roman surface having contact of the fourth order with S at P . Wilczynski's determination* of this surface is the only instance, known to the writer, of any previous study of the contact of a quartic surface with an analytic surface. The method used by Wilczynski gives the following *parametric equations of the osculating Roman surface of S at P* , referred to the tetrahedron, and in the notation, that we are now using:

$$(15) \quad \begin{aligned} x_1 &= 24 - 3(2\beta\psi + \phi^2/8)\xi^2 + (3\phi\psi/4 - 40\beta\gamma/3 - 12\theta_{uv})\xi\eta \\ &\quad - 3(2\gamma\phi + \psi^2/8)\eta^2, \\ x_2 &= 24\xi + 3\phi\xi^2 - 3\psi\xi\eta + 8\gamma\eta^2, \\ x_3 &= 24\eta + 8\beta\xi^2 - 3\phi\xi\eta + 3\psi\eta^2, \\ x_4 &= 24\xi\eta. \end{aligned}$$

5. *Contact of the fifth order.* Proceeding as before, we find that the equation of the most general quartic surface having contact of the fifth order with S at P is

* Wilczynski, "Fifth Memoir," *Transactions of the American Mathematical Society*, Vol. 10 (1909), p. 279. Darboux also determined this surface, *Comptes Rendus*, Vol. 91 (1880), p. 969.

$$\begin{aligned}
 (16) \quad & g [x(z-T) + z(4\beta\psi x^3 + 6\theta_{uv}x^2y + 5\gamma\phi xy^2 - \gamma\psi y^3)/12 \\
 & + \phi x^2(z-xy)/4] + f [y(z-T) + z(-\beta\phi x^3 + 5\beta\psi x^2y \\
 & + 6\theta_{uv}xy^2 + 4\gamma\phi y^3)/12 + \psi y^2(z-xy)/4] \\
 & + h [z-T-U + z\{\beta(c_5-10c_2)x^2y + \gamma(c_6-10c_3)xy^2 \\
 & - 5\beta c_1x^3 - 5\gamma c_4y^3\}/60 + (z-xy)(c_6x^2 + c_5y^2)/20] \\
 & + c [z^2 - x^2y^2 + 2z(\beta x^3 + \gamma y^3)/3] + Cz^4 + Hxz^3 + Iyz^3 \\
 & + Kx^2z^2 + Ly^2z^2 + Mxyz^2 + C'z^3 + z(z-xy)(H'x + I'y) \\
 & + J'[xy(z-xy) + z(\beta x^3 + \gamma y^3)/3] = 0.
 \end{aligned}$$

Since this equation contains fourteen arbitrary homogeneous constants, it follows that there are ∞^{13} quartic surfaces having contact of the fifth order with S at P . Of these there are ∞^9 that are composite, one component of each being the tangent plane $z=0$ and the other being a cubic surface having contact of the third order with S at P . Any one of the ∞^4 other quartic surfaces represented by (14) intersects the tangent plane in a quartic curve whose equations are

$$\begin{aligned}
 (7) \quad & z = g(xT + \phi x^3y/4) + f(yT + \psi xy^3/4) \\
 & + h [T + U + xy(c_6x^2 + c_5y^2)/20] \\
 & + (c + J')x^2y^2 = 0.
 \end{aligned}$$

There are ∞^3 of these curves, which can be discussed as in § IV, the only difference being that when $f=g=h=0$, the curve breaks up into the asymptotic tangents each counted twice.

The fact that two quadrics each of which has contact of the second order constitute a composite quartic with contact of the fifth order suggests that under certain conditions the left member of (16) is factorable into the product of two quadratic expressions of the form

$$z - xy + pxz + qyz + rz^2,$$

since the vanishing of such an expression gives the equation of the most general non-singular quadric having contact of the second order with S at P . In fact, multiplying together two such expressions and comparing the coefficients of the result with the coefficients of (16) we easily show that (16) represents two non-singular quadrics each of which has contact of the second order with S at P if the coefficients of (16) satisfy the conditions.

$$\begin{aligned}
 (18) \quad & f = g = h = 0, \quad c = 1, \quad J' = -2, \\
 & (2H - C'H')^2 = (C'^2 - 4C)(H'^2 - 4K), \\
 & (2I - C'I')^2 = (C'^2 - 4C)(I'^2 - 4L), \\
 & [2(C' + M) - H'I']^2 = (I'^2 - 4L)(H'^2 - 4K).
 \end{aligned}$$

For such a composite quartic, the curve of intersection with the tangent plane is the asymptotic tangents, each counted twice.

6. *Contact of the sixth order.* Since the equation of the most general quartic surface having contact of the sixth order with S at P is rather long, we shall not write it here explicitly, although there would be no difficulty in doing so.* We shall merely write conditions on the coefficients of (16) which are necessary and sufficient that the quartic surface represented by (16) shall have contact of the sixth order. These conditions are found by demanding that (16) be satisfied by the expression for z in (3) identically in x, y as far as the terms of the sixth degree, and can be written in the form

$$\begin{aligned}
 4\beta(c + J') &= \beta\phi f + (12c_0 - 80\beta\psi + 15\phi^2)g/20 \\
 &\quad + (36d_0 + \beta c_1 + 3\phi c_0/20)h, \\
 4\gamma(c + J') &= (12c_5 - 80\gamma\phi + 15\psi^2)f/20 + \gamma\phi g \\
 &\quad + (36d_6 + \gamma c_4 + 3\psi c_5/20)h, \\
 4H' - \phi(2c + J') &= (3c_0 - 25\beta\psi)f/15 + (c_1 - 2\theta_{uv} - \phi\psi)g \\
 &\quad + [12d_1 + (10\beta c_2 - \beta c_5 - 3\phi c_0)/15]h, \\
 4I' - \psi(2c + J') &= (c_4 - 2\theta_{uv} - \phi\psi)f + (3c_5 - 25\gamma\phi)g/15 \\
 &\quad + [12d_5 + (10\gamma c_3 - \gamma c_0 - 3\phi c_5)/15]h, \\
 (19) \quad 12K - 5\beta\psi(2c + J') &= (c_4 - c_1 - 5\phi\psi/4)\beta f + [2\beta(c_5 - 10c_2) \\
 &\quad + 15\phi\theta_{uv}]g/10 + [12\beta(d_5 - d_2) + 3c_0\theta_{uv}/10 \\
 &\quad - \beta\phi c_5/4]h, \\
 12L - 5\gamma\phi(2c + J') &= [2\gamma(c_0 - 10c_3) + 15\psi\theta_{uv}]f/10 \\
 &\quad + (c_1 - c_4 - 5\phi\psi/4)\gamma g + [12\gamma(d_1 - d_4) \\
 &\quad + 3c_5\theta_{uv}/10 - \gamma\psi c_0/4]h, \\
 6\theta_{uv}(2c + J') + 8\beta\gamma(c + J')/3 - 12(C' + M) \\
 &= (2c_2 - \gamma\phi - \psi^2)\beta f + (2c_3 - \beta\psi - \phi^2)\gamma g \\
 &\quad + [12d_3 + (c_1 + c_4)\beta\gamma/3 - (\gamma\phi c_0 + \beta\psi c_5)/5]h.
 \end{aligned}$$

* For details see R. E. McPherson's master's thesis, University of Chicago, 1929.

There are ∞^6 of these quartics, and there are ∞^4 of them that break up into the tangent plane and a cubic surface with the contact of the fourth order. Any one of the ∞^2 others intersects the tangent plane in a quartic curve whose equations are (17) with $c + J'$ satisfying either of the first two of equations (19) while f, g, h satisfy a linear homogeneous equation deduced from these two equations by eliminating $c + J'$. *There are ∞^1 of these curves forming a pencil, and in this pencil there is just one curve which has a triple-point at P ; for it, $h = 0$. Its triple-point tangents are the asymptotic tangents and the covariant line*

$$(20) \quad z = (12c_5 - 100\gamma\phi + 15\psi^2)\beta x + (12c_6 - 100\beta\psi + 15\phi^2)\gamma y = 0.$$

UNIVERSITY OF CHICAGO,
CHICAGO, ILLINOIS.

Non-Monoidal Involutions having a Congruence of Invariant Conics.

BY HAZEL EDITH SCHOONMAKER.

The present paper has for its purpose the derivation of all the birational involutorial point transformations of space which have the following properties:

- (a) each transforms every conic of a linear congruence into itself;
- (b) the transformations cannot be reduced birationally to the monoidal type.

Every conic of a linear congruence, that is, a ∞^2 system such that through each point of space passes just one conic belonging to it, cuts an arbitrary plane in space in a group of an involution of order two which is reducible to the Jonquières, Geiser, or Bertini type, the harmonic homology being included under the Jonquières. We shall study such a linear congruence of class 1* (hence of the Geiser form), whose generating net consists of cubic surfaces, the basis curve of which is a C_7 of genus 5. We shall build such a system and examine its properties.

Then we shall consider those involutorial birational transformations of space under which the conics of the congruence are invariant and derive their equations. Associated with the plane of each conic is a point lying in it and uniquely determined by it, which is the vertex of the pencil of lines joining pairs of conjugate points on the conic. Thus it is the center of a quadratic involution under which the conic in the plane is invariant. The locus of this point may be a point, a curve, or a surface. We shall show in the last two cases, unless the basis curve breaks up or other peculiarities exist, that the congruence cannot be reduced to a bundle of lines, and hence that the transformations are non-monoidal.

In doing this we shall establish the fact that a linear congruence of conics is not reducible to a linear congruence of lines if every directrix curve has an even number of points on a conic of the system.

Next we shall study the forms into which the C_7 may break up, and a

* By the class of a congruence of conics is understood the number of conics in the congruence which have a given line for bisecant.

few cases where other peculiarities exist. In each of these cases we shall derive the equations for the generating surfaces.

Finally we shall mention without discussion other linear congruences which are reducible to the Geiser type. We shall not discuss in detail the Jonquières or Bertini cases, but we shall give some criteria for determining the type to which the involution given by a linear congruence of conics can be reduced.

Some articles by Montesano* suggested this problem. He has developed all the properties of the congruence we have studied, and in parts his treatment is fuller. For instance, he discusses the complex of tangents to the conics, and derives all the elementary characteristics of the system of conics. His treatment, however, is purely synthetic, while much of our problem has been to develop the analytic point of view. Only one paragraph in his first paper is devoted to the transformation upon which we have centered our attention.

I. *Equations of the conics of the congruence.*

1. Consider a bundle of planes and a projective net of quadrics R . Let $O \equiv (0, 0, 0, 1)$ be chosen as the vertex of the bundle of planes. Then any plane of the system has an equation of the form

$$(1) \quad a_1x_1 + a_2x_2 + a_3x_3 = 0.$$

If $H_i = 0$ is the equation of the quadric corresponding to $x_i = 0$, then the net of quadrics is given by

$$(2) \quad a_1H_1 + a_2H_2 + a_3H_3 = 0.$$

Each plane meets the corresponding quadric surface in a conic. Through an arbitrary point (y) passes a single conic of the system $\sum a_ix_i = 0$, $\sum a_iH_i = 0$ the equations of which are given by

$$(3) \quad |xyH(y)| = 0, \quad |H(x)yH(y)| = 0.$$

Since each is uniquely fixed when the point (y) is given, the system $x \sim H$

* D. Montesano: "Su di un sistema lineare di coniche nello spazio," *Torino Atti*, Vol. 27 (1892), pp. 660-690; "Su le congruenze lineari di coniche nello spazio," *Istituto Lombardo Rendiconti*, Ser. 2, Vol. 26 (1893), pp. 589-604; "Su i varii tipi di congruenze lineari di coniche dello spazio; Note I, II," *Napoli Rendiconti*, Ser. 3, Vol. 1 (1895), pp. 93-110, 155-181. A number of later memoirs by him and by others are concerned with the classification of various systems of conics, but do not discuss the birational transformation.

defines a linear congruence of conics. Denote this system of conics by Σ and a conic of the system by γ .

When the plane describes a pencil, the corresponding quadric describes a pencil projective with it, and the conic of intersection generates a cubic surface. Thus $x_1 = px_2$, $H_1 = pH_2$ generates $x_1H_2 - x_2H_1 = 0$. There are three such cubic surfaces forming a net Ξ ,

$$(4) \quad F_i = x_j H_k - x_k H_j \quad (i, j, k = 1, 2, 3).$$

Any cubic surface that can be generated by conics of the system can be expressed linearly in terms of these three. Equation (3a) may now be written

$$(5) \quad F_1(y)x_1 + F_2(y)x_2 + F_3(y)x_3 = 0.$$

2. From the general condition $x_i = kH_i$ and (4) the curve of intersection of any two cubics $F_i = 0$, $F_k = 0$, lies on the third unless $x_m = 0$, $H_m = 0$, which lies on both. That is, any two cubic surfaces of the system intersect in a variable conic, and the residual is a basis curve lying on all the $F_i = 0$ of the system. Since the complete curve of intersection is of order 9, the basis curve is therefore of order 7. Its genus is found to be 5. It will be denoted by C_7 . Since the quadrics of R do not pass through O , the variable conic does not pass through O , yet each surface of the system Ξ does. Hence C_7 passes through O . Any plane through O meets it in 6 points not at O , and this plane also contains the conic associated with it. These six points then must lie on the conic.*

The curve C_7 does not admit a quadrisecant, but its ∞^1 trisecants form a ruled surface θ of order 15 on which C_7 is five-fold. It follows that the surfaces of Ξ passing through any point of a trisecant t of C_7 contain t . Then the variable base of the pencil which these surfaces form is composed of t and a second trisecant t' of C_7 situated in the plane Ot . The trisecants of C_7 then, are distributed in pairs in such a way that the trisecants of one pair compose a degenerate conic of the system Σ and the plane which they determine belongs to O . Any line r of the bundle (O) meets both components of five degenerate conics of Σ . These degenerate conics lie in planes of the pencil (r) of $F: C_7 r$ of Ξ and therefore their planes envelop at O a cone Δ of class 5 and genus 6.

Consider the locus of points common to the pairs of trisecants of C_7

* Every C_7 of genus 5 has just one point O having the property mentioned. R. Sturm: *Synthetische Untersuchungen über Flächen dritter Ordnung*, Leipzig 1867. See p. 229.

which form the degenerate conics of the system. This is a double curve on θ of order 10 and genus 3. It will be denoted by H_{10} .

3. The surfaces $F: C_7$ of the net Ξ cut an arbitrary plane α of space in a net ζ of cubics having in common the points $P_1 \cdots P_7$ in which α cuts C_7 . Therefore all the conics of the system Σ , variable base of a pencil of Ξ , have for their section with α a pair of points which with $P_1 \cdots P_7$ form the base of a pencil of ζ and which form an involution I_a of the eighth order and first class having for fundamental points the points $P_1 \cdots P_7$ as triple points. This is the Geiser involution. The curve of invariant points is a C_6 with the seven fundamental points as double points. This curve which is the locus of the points of contact of the conics of Σ with the plane α evidently contains the points of section of the plane α with H_{10} .

Any space C_7 of genus 5 can be used to obtain a system like the one described in the preceding paragraphs.

It can be shown that through any conic of the system Σ can be passed ∞^4 quadric surfaces, any one of which meets C_7 in 8 points not on the conic. Through these 8 points can be passed a net of quadrics, which being made projective with the given bundle of planes, generates the original system Σ of ∞^2 conics. Thus the C_7 contains an involution I_8 with 4 degrees of freedom.

The Geiser involution in the plane can be generated by the lines of the plane and a projective net of conics.

4. The polar plane of $O \equiv (0, 0, 0, 1)$ as to the quadric belonging to a point (y) , that is, $H \equiv |H(x)yH(y)| = 0$ is simply $\partial H / \partial x_4 = 0$. Thus the polar of O as to γ , the conic determined by (y) is the line o whose equations are

$$\sum F_i(y)x_i = 0, \sum F_i(y)\partial H_i(x)/\partial x_4 = 0.$$

If each H_i were replaced by $H_i + k\sigma x_i$ where $\sigma = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4$, it would give the same line.

If (y) varies, all the polar planes of the system pass through the fixed point O' conjugate to O in the polarity as to the net of quadrics, where O' is given by

$$\partial H_1(x)/\partial x_4 = 0, \partial H_2(x)/\partial x_4 = 0, \partial H_3(x)/\partial x_4 = 0.$$

The projective systems

$$(6) \quad [\partial H_1(x)/\partial x_4]/x_1 = [\partial H_2(x)/\partial x_4]/x_2 = [\partial H_3(x)/\partial x_4]/x_3 = k$$

define a cubic curve C_3 for which every line o is a bisecant.

Now consider the polar quadric of O as to any cubic surface $F_4(x) = 0$ of the system. Since $F_1(x) = x_2H_3 - x_3H_2$, $\partial F_1(x)/\partial x_4 = x_2\partial H_3(x)/\partial x_4 - x_3\partial H_2(x)/\partial x_4$, we may say: given any point (y) , it uniquely determines a conic γ of the system. This conic lies on a cubic F of the system, and the residual section of its plane is a line r passing through O . The polar quadric of the F is therefore the quadric determined by the C_3 and this line r . Thus the net of first polars of the system is exactly the net determined by the cubic curve C_3 . This is seen at once from the system of equations (6).

Let P be the point (y) . There is an F of the net which contains the line OP . Its equation is given by

$$|F(x) \partial F/\partial y_4 \partial_2 F/\partial y_4^2| = 0.$$

If P is on C_3 , the polar quadric is the cone projecting C_7 from O . The cubic surface has a double point at O , and the quadric cone is the tangent cone of the cubic surface at the node. The surface contains 6 lines through O , the tangent at O and the 5 trisecants of C_7 at O . C_3 and C_7 always have the same tangent line at O .

II. The transformations.

5. No birational correspondence exists between two systems in space in which the conics of the system Σ are related to the lines of a bundle. This fact will be established later. There do exist, however, birational correspondences in space in which the correspondence is between two systems of conics Σ . Among these correspondences we shall consider the involutorial ones in which each conic is conjugate to itself.

A correspondence of this kind determines on an arbitrary conic γ of the system Σ an ordinary quadratic involution. Three cases arise: either the center of the involution is the fixed point O , or as γ varies in Σ it describes in a plane ω of the bundle (O) , a curve $C_\mu: O^{\mu-1}$ in such a way that every point G of this curve is related to all the conics γ which have the line OG as a bisecant; or as the conics vary, the point G describes a surface S_m .

6. Case I. Take any point (y) . It defines a conic γ of the congruence uniquely, and it lies in a plane through O . The line $O(y)$ meets γ in a second point (y') . When (y) takes all positions in space, (y') will also. It groups the points in pairs, (y) , (y') in such a way that the line joining any point (y) to its conjugate (y') always passes through the fixed point O . The

involution is therefore monoidal. It will be denoted by I_2 . Under I_2 every γ of the congruence is transformed into itself.

To obtain the equations of the transformation, let (y) be the point determining the conic and (y') a variable point on the line $O(y)$. The coördinates of (y') are then given by

$$(7) \quad y_i' = sy_i, \quad (i = 1, 2, 3), \quad y_4' = r + sy_4.$$

If (y') is a point on the conic its coördinates must satisfy equations (3). The first of these is satisfied identically since the line lies in the plane.

Substitute (7) in (3b). Hereafter we shall use the following notation for the quadric surfaces.

$$H_1(x) \equiv u_0x_4^2 + u_1x_4 + u_2 = 0 \text{ where } u_0 \text{ is a constant,}$$

$$u_1 \equiv \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3,$$

$$u_2 \equiv a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + b_1x_1x_2 + b_2x_1x_3 + b_3x_2x_3.$$

$$H_2(x) \equiv v_0x_4^2 + v_1x_4 + v_2 = 0 \text{ where } v_0 \text{ is a constant,}$$

$$v_1 \equiv \beta_1x_1 + \beta_2x_2 + \beta_3x_3,$$

$$v_2 \equiv c_1x_1^2 + c_2x_2^2 + c_3x_3^2 + d_1x_1x_2 + d_2x_1x_3 + d_3x_2x_3.$$

$$H_3(x) \equiv w_0x_4^2 + w_1x_4 + w_2 = 0 \text{ where } w_0 \text{ is a constant,}$$

$$w_1 \equiv \gamma_1x_1 + \gamma_2x_2 + \gamma_3x_3,$$

$$w_2 \equiv f_1x_1^2 + f_2x_2^2 + f_3x_3^2 + g_1x_1x_2 + g_2x_1x_3 + g_3x_2x_3.$$

Then

$$H_1(y') = s^2H_1(y) + rs(2u_0y_4 + u_1) + u_0r^2,$$

$$H_2(y') = s^2H_2(y) + rs(2v_0y_4 + v_1) + v_0r^2,$$

$$H_3(y') = s^2H_3(y) + rs(2w_0y_4 + w_1) + w_0r^2.$$

After some manipulation we get

$$r = -2y_4^2 |u_0 y_2 w_1| - 2y_4 |u_0 y_2 w_2| - |u_1 y_2 w_2|,$$

$$s = y_4 |u_0 y_2 w_1| + |u_0 y_2 w_2|,$$

in which the determinants are of order 4, 3, or 2. If these are substituted in (7) the coördinates of (y') become

$$\rho y_i' = y_i \quad (i = 1, 2, 3),$$

$$\rho y_4' = -[y_4 |u_0 y_2 w_2| + 2 |u_1 y_2 w_2|] / [y_4 |u_0 y_2 w_1| + |u_0 y_2 w_2|].$$

The transformation is of order 4. The surface of invariant points is

$$y_4^2 |u_0 y_2 w_1| + 2y_4 |u_0 y_2 w_2| + 2 |u_1 y_2 w_2| = 0.$$

This surface can be used to define the transformation for the image of any point P lies on OP and is the harmonic conjugate of P as to the two residual

points in which OP meets this surface. This surface has O for a double point, contains C_7 simply and the trisecants $t_1 \cdots t_5$.

The image of O is $y_4 | u_0 y_2 w_1 | + | u_0 y_2 w_2 | = 0$. The basis curve is defined by

$$y_4 | u_0 y_2 w_2 | + 2 | u_1 y_2 w_2 | = 0, y_4 | u_0 y_2 w_1 | + | u_0 y_2 w_2 | = 0$$

which projects from the cone

$$K_6 = | u_0 y_2 w_2 |^2 - 2 | u_0 y_2 w_1 | | u_1 y_2 w_2 | = 0.$$

The Jacobian is of order 12 and consists of the image of O , taken twice, and this K_6 . The lines $t_1 \cdots t_5$ are fundamental and are those common to the three cones

$$| u_0 y_2 w_1 | = 0, | u_0 y_2 w_2 | = 0, | u_1 y_2 w_2 | = 0.$$

7. Case 2. Each conic of the system Σ is conjugate to itself in an ordinary quadratic involution, the center of which belongs to a plane rational curve $C_\mu: O\mu^{-1}$. Any plane through O meets the curve in one point not at O . Let (ξ) be the point on C_μ in the plane of this conic. The line joining (y) to (ξ) meets the conic in a second point (y') . The relations between (y) and (y') define the involution desired.

The coördinates of any point on the line joining (y) to (ξ) are given by $r\xi + sy$. If this point is to be (y') its coördinates must satisfy equations (3). Substitute in (3b) and expand. Denote by $H(y, C)$ the polar of $H(y)$ as to the point (ξ) on C_μ and $H(C)$ the result of the substitution of the point on C_μ itself. Then the quadric becomes

$$s | H(y, C)y H(y) | + r | H(C)y H(y) | = 0.$$

From this we get

$$s = \sum H_k(C)F_k(y), r = -\sum H_k(y, C)F_k(y).^*$$

The coördinates of y' are therefore

$$(8) \quad \rho y'_i = y_i \sum H_k(C)F_k(y) - \xi_i \sum H_k(y, C)F_k(y) \quad (i = 1 \cdots 4),$$

where ξ_i is still to be determined in terms of (y) .

Let the equations of C_μ in parametric form be

$$(9) \quad \rho \xi_i = R(\lambda)(\lambda - a_i) \quad (i = 1, 2, 3), \quad \rho \xi_4 = f(\lambda)$$

* Here and following the summation is taken from $k = 1$ to $k = 3$.

in which $R(\lambda)$ is a polynomial of order $\mu - 1$ in λ . The curve lies in the plane

$$(10) \quad (a_2 - a_3)x_1 + (a_3 - a_1)x_2 + (a_1 - a_2)x_3 = 0$$

and contains $O \equiv (0, 0, 0, 1)$ as a point of multiplicity $\mu - 1$, the tangents to which are the $\mu - 1$ roots of $R(\lambda) = 0$. The plane associated with a point (y) meets the curve in the point having for its λ the value determined by

$$(11) \quad |(\lambda - a)y H(y)| = 0.$$

From this it follows that $\lambda = \sum a_i F_i(y) / \sum F_i(y)$ and

$$(12) \quad \begin{aligned} \rho \xi_i &= R[\sum a_k F_k(y), \sum F_k(y)] [\sum a_k F_k(y) - a_i \sum F_k(y)] \\ &\quad (i = 1, 2, 3), \\ \rho \xi_4 &= f[\sum a_k F_k(y), \sum F_k(y)]. \end{aligned}$$

On substituting these values in (8), the equations of the transformation are

$$(13) \quad \begin{aligned} \rho y_i' &= y_i \sum H_k(C) F_k(y) - R[\sum a_k F_k(y), \\ &\quad \sum F_k(y)] [\sum a_k F_k(y) - a_i \sum F_k(y)] \sum H_k(y, C) F_k(y) \quad (i = 1, 2, 3), \\ \rho y_4 &= y_4 \sum H_k(C) F_k(y) - \sum H_k(y, C) F_k(y) f[\sum a_k F_k(y), \sum F_k(y)]. \end{aligned}$$

$R(\lambda)(\lambda - a_i)$ and $f(\lambda)$ are of order μ in F_k and F_k are of order 3 in (y) . Therefore they are of order 3μ in (y) . The expression $H_k(y, C)$ is of order $3\mu + 1$, since it is linear in C and in (y) . $H_k(C)$ is of order 6μ in (y) . The transformation (13) is therefore of order $6\mu + 4$.

All of the cubic surfaces $F_k(y) = 0$ pass through C_7 . Since these appear to order μ in the parametric equations of the curve C_μ , it follows that the basis curve C_7 is of multiplicity $2\mu + 1$ on all the surfaces of the web (13).

Now suppose that the point (y) is chosen on C_μ . The plane and the quadric are determined as in the general case, and since the conic determined by their intersection passes through the given point (y) , for this point every point of the conic is a point (y') . Thus C_μ is a fundamental curve, every point of which goes into a conic. Hence C_μ is a double curve on every surface of the web (13).

The plane of C_μ contains a conic of the system. Its equation is (10). This is identical with $(3a')$. Hence for a point (y) on this conic

$$a_2 - a_3 = F_1(y), \quad a_3 - a_1 = F_2(y), \quad a_1 - a_2 = F_3(y).$$

Then $\lambda = 0/0$, and $\lambda - a_i = 0$. Therefore the conic in the plane of C_μ is a 2μ -fold fundamental curve on the web (13). Every point of the conic has the entire conic for image.

Finally, the trisecants of C_7 which meet C_μ apart from O are all fundamental, since C_7 is of multiplicity $2\mu + 1$ and C_μ is double. Since each trisecant meets C_7 in three points we thus have $3(2\mu + 1) + 2$ points on each. Hence these lines lie on every surface of the web and are fundamental. The image of any point is the entire line passing through it. The order of the ruled surface θ was found to be 15. The order of C_μ is μ . Therefore the surface meets the curve in 15μ points. But O is 5-fold on θ_{15} and of multiplicity $\mu - 1$ on C_μ , hence the number of remaining intersections is $15\mu - 5(\mu - 1) = 10\mu + 5$. Therefore there are $5(2\mu + 1)$ simple basis lines, trisecants of C_7 , which meet C_μ on every surface of the web (13).

The images of C_7 and of C_μ together constitute the Jacobian of the web. The order of the Jacobian is $4(6\mu + 3)$; it contains C_7 to multiplicity $4(2\mu + 1) - 1$ and C_μ to multiplicity 7, the conic in the plane of C_μ to multiplicity $4 \cdot 2\mu$ and each trisecant to multiplicity 4. Hence we have

$$\begin{aligned} S_1 &\sim S_{6\mu+4} : C_7^{2\mu+1} C_\mu^2 \gamma^{2\mu} C_\mu t_1 \cdots t_{5(2\mu+1)}, \\ C_7 &\sim \Gamma_{9(2\mu+1)} : C_7^{6\mu+2} C_\mu^6 \gamma^{6\mu} C_\mu t_1^3 \cdots t_{5(2\mu+1)}^3, \\ C_\mu &\sim M_{3(2\mu+1)} : C_7^{2\mu+1} C_\mu \gamma^{2\mu} C_\mu t_1 \cdots t_{5(2\mu+1)}. \end{aligned}$$

To obtain the surface of invariant points, let $y' = y$ in (13). This becomes

$$\sum H_k(y, C) F_k(y) = 0 \text{ which has the signature } K_{2\mu+4} : C_7^{\mu+1} C_\mu \gamma C_\mu t_1 \cdots t_{5(2\mu+1)} H_{10}.$$

The lines joining pairs $(y), (y')$ of conjugate points in the involution are ∞^3 . They consist of all the lines which meet C_μ . Given any line l meeting C_μ at P . Pass a plane through l and O . This plane will meet its associated quadric in a conic containing an involution of conjugate points, the center of which is P . This line in the general position will not contain more than one pair of conjugate points.

An interesting special case occurs when $\mu = 1$. In this case T belongs to a special linear complex.*

8. Case 3. Now suppose the center of the involution under which each γ of Σ remains invariant describes a surface S_m . The lines joining pairs of conjugate points PP' now form a complex. Every point of S_m is the vertex of a pencil, and the plane of our system determined by that point is the plane of the pencil. This is a very particular form of a line complex; ∞^3 .

* M. Pieri: "Sulle trasformazioni birazionali dello spazio inerenti a un complesso lineare speciale," *Palermo Rendiconto*, Vol. 6 (1892), pp. 234-44.

lines arranged in ∞^2 pencils, the vertices describing a surface and the planes all passing through a fixed point O . The planes and the points are in $(1, 1)$ correspondence and in united position, that is, a plane π and its associated point P are perspective, π passes through P .

Conversely, every such complex Γ and the system of planes of Σ determine an involutorial birational transformation which leaves every conic of Σ invariant.

Through an arbitrary point of space passes just one γ of Σ , but through a point of C_7 pass ∞^1 such conics. Let P be such a point of C_7 . There is one F of the net which has P for a double point. All the planes through OP cut from this surface the line OP and a conic of Σ always passing through P but not through O . In each such plane is a point of S_m . The line joining P to this point meets the γ of the plane in one other point P' . The locus of P' is the image of the fundamental point P on C_7 . The point P' must always lie on the $F: P^2$ having the point P for a double point, and every line must lie on the complex cone belonging to P . Let μ be the order of the complex. Every line of the bundle (O) is of multiplicity $\mu - 1$, hence OP is a line of multiplicity $\mu - 1$ on the cone K_μ of P . It is simple line of $F: P^2$, hence the residual curve of intersection is of order $3\mu - (\mu - 1) = 2\mu + 1$. Since P is of multiplicity μ on K_μ and 2 on F , the complete curve has a point of multiplicity 2μ at P . But OP counts for $\mu - 1$, hence the residual curve of order $2\mu + 1$ has $\mu + 1$ branches through P .

9. In the involution I there is now a surface K of invariant or coincident points. It is the locus of points of contact of tangents from points of S_m to the associated conic and contains $C_7^{\mu+1}$. Any conic of Σ has just 2 points on K , not on C_7 , namely the points of contact. Let x be the order of K . A conic of Σ meets C_7 in six points, and each counts for $\mu + 1$ intersections with K , hence the $2x$ intersections of γ and K are made up of $6(\mu + 1) + 2$. Therefore $x = 3\mu + 4$.

10. In an arbitrary plane α of space the pairs of distinct points of the involution I are conjugate in the Geiser involution determined by the points in which the plane α meets C_7 , and the lines joining such pairs of conjugate points all belong to the complex Γ ; that is, they envelop a curve of class μ . We have seen that when a point describes a straight line r , the lines from O to the conjugate describe a cone of class 3. Therefore if r is in the plane α , these lines form an envelop of class 3. Hence the points in α whose conjugates also lie in α form a curve of order 3μ . Moreover, α cuts K in a curve of order $3\mu + 4$. These two curves together form the complete intersection

of α and its conjugate surface in I . Hence the order of the conjugate surface, the order of the involution, is $6\mu + 4$.

11. Let r be a line passing through O . A plane through r contains an associated point of S_m . As the plane turns about r , the point describes a curve such that each position of the plane contains just one point of it, apart from points on r . Let Q be any fixed point of r . In every position of the plane connect Q with (ξ) , the corresponding point of S_m in this plane. This line $Q(\xi)$ meets a pair of conjugate points on the conic of its plane, hence it is a line of the complex. The curve must then lie on the complex cone of Q and meet every generator once, hence the locus of the points on S_m is of order μ , and it therefore meets r in $\mu - 1$ points.

Consider any plane section π of S_m . To each of its points P corresponds a plane of the bundle (O) and passing through P . The lines of the pencil having P for vertex and lying in the associated plane through O all belong to the complex Γ . One of these lines lies in the given plane π . Conversely, all the lines of Γ in π are in the corresponding planes.

If each of these lines of Γ in π is connected with O , the planes envelop a cone of class μ . Some of the planes may contain fundamental curves, to which the plane is associated rather than to a point. Let $\sigma_1 \dots \sigma_h$ be such planes belonging to K_μ , and let σ_i be counted to multiplicity δ_i . Moreover, let S_i be the order of the fundamental curve in σ_i . Then every line of the plane σ_i is a multiple line of Γ to multiplicity s_i . The conic of Σ is invariant under ∞^1 involutions, namely, all those having a point of C_{s_i} for center. This requires that at every point in which γ meets C_{s_i} the point be fundamental, corresponding to the whole conic. Each such conic is therefore a $2s_i$ -fold fundamental curve of the second kind.

An arbitrary line r meets S_m in m points. These are associated with m planes of (O) which belong to all the cones, images of all the plane sections through r . Hence the two K_μ, K'_μ , having μ^2 common tangent planes, have also the fundamental planes σ_i , hence $\mu^2 - \sum \delta_i^2 = m$.

12. There are ∞^2 conics in Σ . Among them are ∞^1 , each passing through the point of S_m associated with its plane. Such conics are images of the associated points of S_m lying on them. The locus of such points is a fundamental curve, double for the surfaces, images of the planes of space in I , since the image of each point is a curve of the second order. Let ν be the order of this curve C_ν . It is of the first kind. It meets every fundamental γ of the second kind in the $2s_i$ points in which γ is met by the curve C_{s_i} , locus of associated points of its plane. Therefore the planes of the conics containing

points of S_m on the conic envelop a cone containing the singular planes σ_i to multiplicity $2s_i$. The class of this cone is $2\mu + 1$.

From this cone and that of class 5, the planes of which contain composite conics, we obtain the result that there are $5(2\mu + 1)$ planes through O which contain composite conics—that is, a pair of lines, and the associated point of S_m lies on one of them. There are consequently $5(2\mu + 1)$ lines t_i such that in I any point of any one of them has the entire line through it for image.

13. This completes the list of fundamental lines in the involution, hence we may write

$$S_1 \sim S_{6\mu+4} : C_7^{2\mu+1} C_v^2 : \gamma_1^{2s_1} \cdots \gamma_h^{2s_h} t_1 \cdots t_{5(2\mu+1)}.$$

Since the image of an arbitrary straight line is a curve of order $6\mu + 4$, this means that the variable curve of any two $S_{6\mu+4}$ of the web has all its intersections in the basis or fundamental curves, that is,

$$(6\mu + 4)^2 - 7(2\mu + 1)^2 - 4\nu - 8 \sum \delta^2 - 5(2\mu + 1) = 6\mu + 4, \text{ hence} \\ \nu = 2\mu^2 + \mu - 2 \sum \delta^2 = 2m + \mu.$$

$$C_v \sim \Gamma_{3(2\mu+1)} : C_7^{2\mu+1} C_v \gamma_1^{2s_1} \cdots \gamma_h^{2s_h} t_1 \cdots t_{5(2\mu+1)},$$

$$C_7 \sim M_{9(2\mu+1)} : C_7^{6\mu+2} C_v^6 \gamma_1^{6s_1} \cdots \gamma_h^{6s_h} t_1^3 \cdots t_{5(2\mu+1)}^3.$$

The other fundamental curves are all of the second kind, each point having for image the entire curve through it. The Jacobian of the web $S_{6\mu+4}$ is of order $24\mu + 12$. It consists of $\Gamma_{3(2\mu+1)}$ and $M_{9(2\mu+1)}$. The surface of invariant points is

$$K_{3\mu+4} : C_7^{\mu+1} C_v \gamma_1^{s_1} \cdots \gamma_h^{s_h} t_1 \cdots t_{5(2\mu+1)}.$$

All the lines t_i are trisecants of C_7 . The surface $K_{3\mu+4}$ of invariant points also contain H_{10} , double on θ_{15} for the plane of a pair intersecting at any point of H_{10} passes through O , and every γ of Σ remains invariant under I .

The numbers m, μ, s_i, h, δ_i are connected by the single relation

$$\mu^2 - \sum \delta_i^2 = m, \text{ hence } \mu \geq m.$$

14. Analytically, proceeding as in Case 2 we get as analogous to (8)

$$(14) \quad \rho y_i' = y_i \sum H_k(S_m) F_k(y) - \xi_i \sum H_k(y, S_m) F_k(y) \\ (i = 1, 2, 3, 4).$$

where ξ_i denotes a point on the surface S_m still to be determined in terms of y ; $H(y, S_m)$ the polar of $H(y)$ as to the point (ξ) on S_m ; $H(S_m)$ the result of the substitution of (ξ) itself.

Let us consider the following cases:

(a) Let the surface be a plane not passing through O . Let the equation of S_m be $x_4 = 0$. The coordinates of any point on S_m are $\xi \equiv (\xi_1, \xi_2, \xi_3, 0)$. Transform (ξ) into (ξ') by means of

$$\xi'_i = a_i \xi_i, \quad (i = 1, 2, 3), \quad \xi'_4 = 0.$$

where $a_i \neq 0$ and all are distinct. This establishes a (1, 1) correspondence between the points of S_m and the planes passing through these points. Hence if the plane of the bundle is so taken as to contain one of these lines, the required conditions are satisfied. The plane through O , (ξ) , (ξ') is

$$|x \xi a \xi| = 0,$$

which becomes

$$(a_3 - a_2)\xi_3\xi_2x_1 + (a_1 - a_3)\xi_1\xi_3x_2 + (a_2 - a_1)\xi_2\xi_1x_3 = 0.$$

Since this is a plane of the bundle it is identical with (5). From this it follows that

$$(15) \quad F_1(y) = \xi_3\xi_2(a_3 - a_2), \quad F_2(y) = \xi_1\xi_3(a_1 - a_3), \quad F_3(y) = \xi_2\xi_1(a_2 - a_1).$$

Upon solving, we get

$$(16) \quad \rho\xi_1 = F_3F_2(a_3 - a_2), \quad \rho\xi_2 = F_1F_3(a_1 - a_3), \quad \rho\xi_3 = F_2F_1(a_2 - a_1).$$

When these values are substituted in (14) the equations of the transformation are given by

$$(17) \quad \begin{aligned} \rho y_1' &= y_1 \sum H_k(S_m)F_k(y) - (a_3 - a_2)F_3F_2 \sum H_k(y, S_m)F_k(y), \\ \rho y_2' &= y_2 \sum H_k(S_m)F_k(y) - (a_1 - a_3)F_1F_3 \sum H_k(y, S_m)F_k(y), \\ \rho y_3' &= y_3 \sum H_k(S_m)F_k(y) - (a_2 - a_1)F_2F_1 \sum H_k(y, S_m)F_k(y), \\ \rho y_4' &= y_4 \sum H_k(S_m)F_k(y). \end{aligned}$$

Since (ξ) is of degree 6 in (y) , $H(y, S_m)$ of degree 7 in (y) , and $H(S_m)$ of degree 12, the transformation is of order 16. Therefore $6\mu + 4 = 16$ and $\mu = 2$.

The vertices of the triangle of reference in the plane $x_4 = 0$ are singular points. The image of each of these points is the whole pencil of lines having the point for vertex. Therefore we have three bundles of lines, apart from O , that are singular lines of the complex.

All the cubic surfaces $F_k = 0$ pass through C_7 . Since these appear to order 4 on $H(S_m)$ and to order 2 on $H_k(y, S_m)$, it follows that C_7 is of multiplicity 5 on the image.

15. Consider a pencil of planes through the line $Q(\xi)$. In $x_4 = 0$

each line of the pencil has a point associated with it, and for one position of the line the associated point is (ξ) itself. Hence the locus is a conic. Now take any point P in space. The line OP meets $x_4 = 0$ in (ξ) . The complex cone belonging to P contains the conic in $x_4 = 0$. This again proves that the complex is quadratic, and that the bundle (O) belongs to it simply.

The point (ξ) is associated with the line

$$(a_2 - a_3)\xi_2\xi_3x_1 + (a_3 - a_1)\xi_3\xi_1x_2 + (a_1 - a_2)\xi_1\xi_2x_3 = 0$$

in $x_4 = 0$, and hence with the plane of (O) having the same equation. The quadric of the net corresponding to the latter has the equation

$$(a_2 - a_3)\xi_2\xi_3H_1(x) + (a_3 - a_1)\xi_3\xi_1H_2(x) + (a_1 - a_2)\xi_1\xi_2H_3(x) = 0.$$

The point (ξ) lies on the associated conic of Σ in its plane if it lies on the quadric. Therefore

$$(a_2 - a_3)\xi_2\xi_3H_1(\xi) + (a_3 - a_1)\xi_3\xi_1H_2(\xi) + (a_1 - a_2)\xi_1\xi_2H_3(x) = 0$$

is the equation of the C_4 in $x_4 = 0$, every point of which has the whole conic of Σ passing through it for conjugate in I . Hence C_4 is double on the conjugate of every plane in I . This C_4 passes through the fundamental points in $x_4 = 0$ simply.

The planes containing the sides of the triangle of reference in the x_4 -plane have a fundamental conic double.

Finally, for certain values of (ξ) , the point (ξ) lies on the associated conic of Σ , and the latter is composite. The first condition requires that (ξ) lie on C_4 : the second that the plane containing (ξ) shall be tangent to the associated quadric. This gives rise to a determinant of order 5, each element of which is quadratic in (ξ) , and therefore represents a C_{10} which meets C_4 in 40 points. Each fundamental point counts for 5 intersections, and therefore there are 25 fundamental trisecants of C_7 .

We have then

$$\begin{aligned} S_1 \sim S_{16} &: C_7^5 C_4^2 \gamma_1^2 \gamma_2^2 \gamma_3^2 t_1 \cdots t_{25}, \\ C_4 \sim \Gamma_{15} &: C_7^5 C_4 \gamma_1^2 \gamma_2^2 \gamma_3^2 t_1 \cdots t_{25}, \\ C_7 \sim M_{45} &: C_7^{14} C_4^6 \gamma_1^6 \gamma_2^6 \gamma_3^6 t_1^3 \cdots t_{25}^3. \end{aligned}$$

The surface of invariant points is obtained by letting $y = \gamma'$ in (17). This gives

$$K_{10} = C_7^3 C_4 \gamma_1 \gamma_2 \gamma_3 t_1 \cdots t_{25} H_{10}.$$

16. (b) Let the surface S_m be a plane passing through O . Let its equation be $x_1 = 0$. Then in the correspondence between the lines passing

through points on S_m and the planes of the bundle, it is necessary that the lines themselves pass through O . This correspondence can be established as follows. Let $\sum a_i x_i = 0$ be any fixed plane through O . Then

$$\lambda(a_2 x_2 + a_3 x_3) + \mu x_1 = 0$$

defines a pencil of planes, all passing through $x_1 = 0$, $\sum a_i x_i = 0$. Make these planes projective with points on the line, in such a way that $x_1 = 0$ is associated with O for every set of a_i , and the plane $a_2 x_2 + a_3 x_3 = 0$ is associated with the point $O' \equiv (0, a_3, -a_2, 0)$. We get then

$$\lambda(0, a_3, -a_2, 0) + \mu(0, 0, 0, 1) = 0.$$

If $\lambda = \mu$, $x_1 + a_2 x_2 + a_3 x_3 = 0$ is associated with $\xi \equiv (0, a_3, -a_2, 1)$. This is a plane of the bundle of $(3a')$. We get then

$$\rho \xi_1 = 0, \quad \rho \xi_2 = F_3(y), \quad \rho \xi_3 = -F_2(y), \quad \rho \xi_4 = F_1(y).$$

Substituting these values in (14) which applies to this case also, gives as the equations of the transformation

$$\begin{aligned} \rho y'_1 &= y_1 \sum H_k(S_m) F_k(y), \\ (18) \quad \rho y'_2 &= y_2 \sum H_k(S_m) F_k(y) - F_3(y) \sum H_k(y, S_m) F_k(y), \\ \rho y'_3 &= y_3 \sum H_k(S_m) F_k(y) + F_2(y) \sum H_k(y, S_m) F_k(y), \\ \rho y'_4 &= y_4 \sum H_k(S_m) F_k(y) - F_1(y) \sum H_k(y, S_m) F_k(y), \end{aligned}$$

The transformation is of order 10 and $\mu = 1$. The multiplicity of C_7 is 3 on the image. The point (ξ) is associated with the plane

$$\xi_4 x_1 - \xi_3 x_2 + \xi_2 x_3 = 0.$$

The equation of C_3 is then,

$$\xi_4 H_1(\xi) - \xi_3 H_2(\xi) + \xi_2 H_3(\xi) = 0.$$

We have then

$$\begin{aligned} S_1 \sim S_{10} &: C_7^3 C_3^2 t_1 \cdots t_{15}, \\ C_3 \sim \Gamma_9 &: C_7^3 C_3 t_1 \cdots t_{15}, \\ C_7 \sim M_{27} &: C_7^8 C_3^6 t_1^3 \cdots t_{15}^3. \end{aligned}$$

The surface of invariant points is

$$\sum H_k(y, S_m) F_k(y) = 0 \text{ of form } C_7^2 C_3 t_1 \cdots t_{15} H_{10}.*$$

* This is included as a particular case of the transformation belonging to a linear complex. D. Montesano: "Su le trasformazioni involutorie dello spazio che determinano un complesso lineare di rette," *Accademia dei Lincei Rendiconti*, Ser. 4, Vol. 4 (1888), pp. 107-16; 277-86.

17. (c) Let S_m be a quadric passing through O . One method of relating the planes of the bundle with the points of S_m is as follows. Let $S_m \equiv x_1x_4 - x_2x_3 = 0$. Let (ξ) be a point on S_m . Project (ξ) from O on the plane $x_4 = 0$. The coordinates of (ξ') are $\xi_1, \xi_2, \xi_3, 0$. Make a transformation in the plane $x_4 = 0$ such that $\xi_i'' = a_i\xi_i'$, a_i distinct and $\neq 0$. Take as the plane of the bundle, the plane containing O , (ξ') , (ξ'') . This has for its equation

$$(a_3 - a_2)\xi_3\xi_2x_1 + (a_1 - a_3)\xi_1\xi_3x_2 + (a_2 - a_1)\xi_2\xi_1x_3 = 0.$$

From this we get equations (15). These and the one obtained from the fact that (ξ) lies on the surface give

$$\begin{aligned}\rho\xi_1 &= F_3F_2(a_3 - a_2), & \rho\xi_2 &= F_1F_3(a_1 - a_3), & \rho\xi_3 &= F_2F_1(a_2 - a_1), \\ \rho\xi_4 &= [(a_1 - a_3)(a_2 - a_1)/(a_3 - a_2)] F_1^2.\end{aligned}$$

18. Another method is as follows: take the surface as before. It contains the line l whose equations are $x_1 = x_2$, $x_3 = x_4$. It does not contain the line l' , the equations of which are $x_1 = 0$, $x_4 = 0$. The coordinates of points on l' will be denoted by (z) . Through (ξ) passes only one transversal of these two lines.

Relate (ξ) to the transversal passing through it, and take as the plane of the bundle, the plane through O containing this transversal. To set up this relation, take the plane through (ξ) and two points on l . Its equation is

$$(24) + (41) + (13) + (32) = 0, \quad [(ij) = x_i\xi_j - x_j\xi_i].$$

This will cut l' in the point (z) whose coordinates are $z_1 = z_4 = 0$ and

$$z_2 = \xi_1 - \xi_2, \quad z_3 = \xi_4 - \xi_3.$$

Let the plane of the bundle be the plane through O , (ξ) , (z) . Its equation reduces to

$$(\xi_2\xi_4 - \xi_1\xi_3)x_1 + (-\xi_1\xi_4 + \xi_1\xi_3)x_2 + (\xi_1^2 - \xi_1\xi_2)x_3 = 0.$$

This gives

$$F_1 = \xi_2\xi_4 - \xi_1\xi_3, \quad F_2 = -\xi_1\xi_4 + \xi_1\xi_3, \quad F_3 = \xi_1^2 - \xi_1\xi_2.$$

and since (ξ) is on the surface, $\xi_4 = \xi_2\xi_3/\xi_1$. By solving these four equations, we get

$$\rho\xi_1 = F_2F_3, \quad \rho\xi_2 = -F_3(F_1 + F_2), \quad \rho\xi_3 = F_2^2, \quad \rho\xi_4 = -F_2(F_1 + F_2).$$

This is the same result as for the previous method if

$$a_3 - a_2 = 1, \quad a_1 - a_3 = -(F_1 + F_2)/F_1, \quad a_2 - a_1 = F_2/F_1.$$

Substituting these results in (14), gives as the equations of the transformation

$$\begin{aligned} \rho y_1' &= y_1 \sum H_k(S_m) F_k(y) - (a_3 - a_2) F_3 F_2 \sum H_k(y, S_m) F_k(y), \\ \rho y_2' &= y_2 \sum H_k(S_m) F_k(y) - (a_1 - a_3) F_1 F_3 \sum H_k(y, S_m) F_k(y), \\ \rho y_3' &= y_3 \sum H_k(S_m) F_k(y) - (a_2 - a_1) F_2 F_1 \sum H_k(y, S_m) F_k(y), \\ \rho y_4' &= y_4 \sum H_k(S_m) F_k(y) \\ &\quad - [a_1 - a_3] (a_2 - a_1) / (a_3 - a_2) F_1^2(y) \sum H_k(y, S_m) F_k(y). \end{aligned}$$

The transformation is of order 16 and $\mu = 2$. The image contains C_7 to multiplicity 5, 25 trisecants, C_6^2 , and 2 fundamental conics double. In the second method these conics lie in the planes, one of which contains l , and the other l' . The lines of the bundle (O) are simple for through O there is one transversal of l and l' and through that transversal there is a pencil of planes to which O is therefore related.

19. (d) Let S_m be a quadric not passing through O . Using the second method, let the equation of S_m be $x_1 x_3 - x_2^2 + x_4^2 = 0$. It contains the line l whose equations are $x_1 = x_2 - x_4$, $x_3 = x_2 + x_4$. It does not contain the line l' the equations of which are $x_1 = 0$, $x_4 = 0$. Proceeding exactly as in the previous case, we find that

$$z_2 = \xi_2 - \xi_4 - \xi_1, \quad z_3 = \xi_3 - 2\xi_4 - \xi_1.$$

The plane of the bundle reduces to

$$(\xi_2 z_3 - \xi_3 z_2) x_1 - \xi_1 z_3 x_2 + \xi_1 z_2 x_3 = 0.$$

We get

$$\begin{aligned} F_1 &= \xi_1 \xi_2 - \xi_1 \xi_3 - \xi_3 \xi_4 + 2\xi_2 \xi_4, & F_2 &= \xi_1 \xi_3 - 2\xi_1 \xi_4 - \xi_1^2, \\ F_3 &= -\xi_1 \xi_2 + \xi_1 \xi_4 + \xi_1^2. \end{aligned}$$

Solving these together with the relation obtained from the fact that (ξ) lies on the quadric, gives

$$\begin{aligned} \rho \xi_1 &= F_3^2 (F_2 + 2F_3), & \rho \xi_2 &= -F_3 (F_2^2 + 2F_2 F_3 - F_1 F_3 + F_3^2), \\ \rho \xi_3 &= (F_2 + F_3) (F_2^2 + F_2 F_3 - 2F_1 F_3), & \rho \xi_4 &= -F_3^2 (F_1 + F_2 + F_3). \end{aligned}$$

Substituting these in (14) gives the equations of the transformation which are of degree 22 in y . Therefore $\mu = 3$.

20. (e) Humbert* discusses a remarkable complex of conics by considering the doubly infinite system of space cubic curves which pass through 5 fixed points in space, namely, the four vertices of the tetrahedron of reference and the unit point. The locus of the point of contact of those cubics of the system which touch a given plane gives a conic of the system. These conics satisfy the two equations

$$\lambda_1 x + \lambda_2 y + \lambda_3 z + \lambda_4 t = 0, \quad \lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + \lambda_4 t^2 = 0.$$

The tangents to the space cubics form a complex of the sixth order. It follows that the locus of the points of contact of the tangents which can be drawn through any point P in space to the cubics is a C_7 of genus 5 which passes simply through P and the 5 fundamental points and has 6 points on every conic in any plane through P . The point P is called the pole of the C_7 .

The C_7 is situated on 5 cones of the third order whose vertices are the 5 fundamental points, and all of whose generators cut C_7 in 2 variable points, that is, they are trisecants of C_7 . The curves of the system $|C_7|$ also pass through the ten points which are the intersections of the straight lines joining two fundamental points with the plane containing the other three fundamental points. These curves lie on a doubly infinite linear system of cubic surfaces. There are a triple infinity of them, one for each point in space.

By considering only one bundle of planes this system reduces to a special case of the one under discussion. The vertex of the bundle is the pole of C_7 . Let the pole be the point $0 \equiv (a_1, a_2, a_3, a_4)$. On changing the notation the equations determining the conics become

$$\begin{aligned} \lambda_1(a_1x_4 - a_4x_1) + \lambda_2(a_2x_4 - a_4x_2) + \lambda_3(a_3x_4 - a_4x_3) &= 0, \\ \lambda_1(a_1x_4^2 - a_4x_1^2) + \lambda_2(a_2x_4^2 - a_4x_2^2) + \lambda_3(a_3x_4^2 - a_4x_3^2) &= 0. \end{aligned}$$

By means of the transformation

$$x_i = a_i y_i + y_i, \quad (i = 1, 2, 3), \quad x_4 = a_4 y_4$$

the equations of the conics become

$$\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 = 0, \quad \lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3 = 0$$

where

$$H_i = y_i^2(a_i a_4 - a_i^2) - 2a_i y_i y_4 - y_i^2 = 0.$$

* G. Humbert: "Sur un complexe remarquable de coniques," *Journal de l'école polytechnique*, Vol. 64 (1893), pp. 123-149.

The fundamental points are transformed as follows:

$$\begin{aligned}(0, 0, 0, 1) &\sim (a_1, a_2, a_3, -1), \\(1, 0, 0, 0) &\sim (1, 0, 0, 0), \\(0, 1, 0, 0) &\sim (0, 1, 0, 0), \\(0, 0, 1, 0) &\sim (0, 0, 1, 0), \\(1, 1, 1, 1) &\sim (a_4 - a_1, a_4 - a_2, a_4 - a_3, 1).\end{aligned}$$

F_1, F_2, F_3 are cubic cones with vertices at $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(1, 0, 0, 0)$ respectively. The surface of trisecants breaks up into 5 cubic cones, and H_{10} becomes the 10 lines joining the 5 fundamental points by two's.

III. Reducibility of a linear congruence of conics to a linear congruence of lines.

21. If a linear congruence of conics has a directrix which has an odd number of points on each conic of the system, this congruence is reducible to a linear congruence of lines by means of a birational transformation.

22. Case 1. If a directrix has one point on each conic of Σ , a correspondence can be established in the following manner. Relate the conics of Σ birationally to the lines of a linear congruence Σ' , add a pencil of lines $(A - \alpha)$ to a pencil of planes (a') . Any point P in space determines a conic γ in the plane ω . This conic passes through P , has one point D on the directrix, and corresponds to the line g' of the congruence Σ' . PD cuts one line of the pencil $(A - \alpha)$, namely the line r , determined by the point in which PD cuts the line of intersection of the planes ω and α . The line r corresponds to the plane ρ' of (a') . This plane ρ' is cut by the line g' in a point which is P' , the desired conjugate of P .

23. Case 2. If a directrix has three points on a conic of Σ , the correspondence is established in the following way. Relate the conics of Σ to the lines of a congruence Σ' , and the pairs of an ordinary involution I of lines in a pencil $(A - \alpha)$ to the planes of a pencil (a') . For any point P in space consider the conic of Σ which corresponds to the line g' of Σ' . Through P and the three points in which the directrix cuts γ , passes a pencil of conics cutting α in two points. For only one conic γ' , do these two points lie on conjugate lines r and r' of the involution I . These lines r and r' correspond to a plane ρ' of the pencil (a') . The plane ρ' is cut by g' in a point P' .

24. Case 3. If the congruence Σ has a directrix which has five points

on each γ , since the total number of such points is six, then the congruence has a directrix with one point on γ and this reduces to Case 1. In all three cases the conics of Σ are evidently related to the lines of Σ' in a (1,1) correspondence.

25. Conversely, a linear congruence of conics is not reducible by means of a birational transformation in space to a linear congruence of lines if it has only one directrix, or if it has two which have four and two points respectively on each conic of Σ , or if it has three, each of which cuts γ in two points.

In all three cases it is not possible to relate to a conic of the congruence Σ one of its points, for if it were possible, the surface which is the locus of such points would have in common with each conic of the system Σ , aside from the directrices, the one point related to γ . This is absurd for the points common to γ and the directrices consist of an even number (or zero) points of section. Therefore in these cases there is no birational transformation in space making the conics of Σ correspond to the lines of a linear congruence Σ' . If this correspondence existed, to a plane of the system Σ' would correspond in the other system a surface having apart from the directrices, one point on every conic of Σ .

26. It is true, however, that a linear congruence of conics which has an exceptional point is reducible by means of a birational transformation in space, to a linear congruence of lines. The proof of this is entirely analogous to that given for the congruence when the conics have only one point on a directrix. It is merely necessary in the argument to substitute the exceptional point of the congruence for the single point of section of the indicated directrix.

IV. *The cases for which the C_7 is composite.**

27. Case 1. If each cubic surface of the net contains a line not passing through O , the C_7 consists of this line which is a trisecant, and of a C_6 of genus 3.

Let the line t be the line whose equations are $x_1 = 0, x_4 = 0$. The equations of the cubic surfaces are given by (5), where as before

$$H_1 = u_0 x_4^2 + u_1 x_4 + u_2 = 0, \text{ etc.}$$

* Montesano enumerates these, but without discussion or equations. "Su i varii tipi di congruenze lineari di coniche dello spazio; Note II," *Napoli Rendiconti*, Ser. 3, Vol. 1 (1895).

If t is on the cubic surfaces the quadrics of the net R are of the form

$$H_1 \equiv u_0 x_4^2 + u_1 x_4 + (a_1 x_1 + b_1 x_2 + b_2 x_3) x_1 = 0,$$

$$H_2 \equiv v_0 x_4^2 + v_1 x_4 + (c_1 x_1 + d_1 x_2 + d_2 x_3) x_1 + (c_2 x_2 + d_3 x_3) x_2 = 0,$$

$$H_3 \equiv w_0 x_4^2 + w_1 x_4 + (f_1 x_1 + g_1 x_2 + g_2 x_3) x_1 + (c_2 x_2 + d_3 x_3) x_3 = 0.$$

If $u_1 = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$ as before, the conic $x_1 = 0$, $H_1 = 0$ passes through $X \equiv (0, -\alpha_3, \alpha_2, 0)$. Similarly the conics of the system in the planes $x_2 = 0$, $x_3 = 0$ pass through $Y \equiv (0, 0, 1, 0)$; $Y' \equiv (-\alpha_2, 0, c_1, 0)$ and $Z \equiv (0, 1, 0, 0)$; $Z' \equiv (g_1, -f_1, 0, 0)$ respectively. The section of F_1 by the plane $x_4 = 0$ consists of t and a conic passing through Y' and Z' . The section of F_2 by the same plane consists of t and a conic passing through Z' . These two conics have three points of intersection apart from C_7 . It follows that t is a trisecant of C_6 which is therefore of genus 3.

28. Case 2. Let each cubic surface of the net contain a line passing through O . Then this line is a quadrisecant and the C_6 is of genus 2.

Let the equations of the line q be $x_1 = 0$, $x_2 = 0$. If F_1 and F_2 contain q , then H_1 and H_2 must contain q . The third quadric is perfectly arbitrary. These quadrics are then of the form

$$H_1 \equiv (\alpha_1 x_1 + \alpha_2 x_2) x_4 + (a_1 x_1 + b_1 x_2 + b_2 x_3) x_1 + a_2 x_2^2 = 0,$$

$$H_2 \equiv (\beta_1 x_1 + \beta_2 x_2) x_4 + (c_1 x_1 + d_1 x_2 + d_2 x_3) x_1 + c_2 x_2^2 = 0,$$

$$H_3 \equiv w_0 x_4^2 + w_1 x_4 + w_2 = 0.$$

The point O is simple on each cubic surface and lies on q . The residual intersection of any plane through q with any two cubic surfaces consists of two conics which have two points in common apart from q . The line q is therefore a quadrisecant and the genus of C_6 is 2.

29. Case 3. Let every cubic surface of the net contain a conic not passing through O . Let the conic C_2 have for its equations $x_1 x_2 - x_3^2 = 0$, $x_4 = 0$. Then the quadrics are of the form

$$H_1 \equiv u_0 x_4^2 + u_1 x_4 + a_1 x_1^2 + a_3 x_3^2 + b_1 x_1 x_2 + b_2 x_1 x_3 = 0,$$

$$H_2 \equiv v_0 x_4^2 + v_1 x_4 + (a_3 + b_1) x_2^2 + c_3 x_3^2 + (a_1 - c_3) x_1 x_2 + b_2 x_2 x_3 = 0,$$

$$H_3 \equiv w_0 x_4^2 + w_1 x_4 + (b_2 - g_1) x_3^2 + g_1 x_1 x_2 + a_1 x_1 x_3 + (a_3 + b_1) x_2 x_3 = 0.$$

The conic $H_1 = 0$, $x_1 = 0$ is tangent to the plane $x_4 = 0$ at the point $(0, 1, 0, 0)$. The conic $H_2 = 0$, $x_2 = 0$ is tangent to the same plane at the point $(1, 0, 0, 0)$. The conic $H_3 = 0$, $x_3 = 0$ passes through the points

$(0, 1, 0, 0)$ and $(1, 0, 0, 0)$. Therefore C_2 has four points on C_5 which is then of genus 2.

30. Case 4. Let the cubic surfaces of the net contain a conic passing through O . Let the equations of this conic be $x_1x_4 - x_3^2 = 0$, $x_2 = 0$. Then the quadrics are

$$\begin{aligned} H_1 &\equiv u_1x_4 + a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + b_1x_1x_2 + b_2x_1x_3 + b_2x_2x_3 = 0, \\ H_2 &\equiv (\beta_1x_1 + \beta_2x_2)x_4 + c_2x_2^2 - b_1x_3^2 + d_1x_1x_2 + d_3x_2x_3 = 0, \\ H_3 &\equiv \alpha_3x_4^2 + [(b_2 - f_3)x_1 + \gamma_2x_2 + (\alpha_1 + a_3)x_3]x_4 + f_2x_2^2 + f_3x_3^2 \\ &\quad + g_1x_1x_2 + g_3x_2x_3 + a_1x_1x_3 = 0. \end{aligned}$$

The residual basis curve is a C_5 which does not pass through O . This C_5 has 5 points in the plane $x_2 = 0$ which must be on C_2 since C_2 is the only curve in the plane common to all the cubic surfaces of the net. Therefore C_5 has genus 1.

31. Case 5. Let each cubic surface of the net contain a space cubic not passing through O . Let the given C_3 be the residual intersection of

$$\begin{aligned} H &\equiv x_1x_2 - x_3x_4 = 0 \text{ and} \\ H' &\equiv m_0x_4^2 + (m_1x_1 + m_2x_2 + m_3x_3)x_4 + (\mu_1x_1 + \mu_2x_2 + \mu_3x_3)x_1 = 0. \end{aligned}$$

Eliminate x_4 and remove the factor x_1 giving

$$m_0x_1x_2^2 + (m_1x_2x_1 + m_2x_2^2)x_3 + (m_3x_2 + \mu_1x_1 + \mu_2x_2)x_3^2 + \mu_3x_3^3 = 0$$

as the equation of the cone projecting C_3 from O . Eliminate x_4 between F_1 and H which gives the cone projecting the curve of intersection from O . Impose the necessary conditions that this shall contain the above cone. Do likewise for F_2 and F_3 and the quadrics become

$$\begin{aligned} H_1 &\equiv (\alpha_1x_1 + \alpha_3x_3)x_4 + a_1x_1^2 + b_2x_1x_3 = 0, \\ H_2 &\equiv [\alpha_1x_2 + (a_1 - d_1)x_3]x_4 + \alpha_3x_2^2 + d_1x_1x_2 + b_2x_2x_3 = 0, \\ H_3 &\equiv m_0x_4^2 + [m_1x_1 + m_2x_2 + (m_3 + \mu_2 + \alpha_1 - g_1)x_3]x_4 + \mu_1x_1^2 \\ &\quad + b_2x_3^2 + g_1x_1x_2 + (\mu_3 + a_1)x_1x_3 + \alpha_3x_2x_3 = 0. \end{aligned}$$

The C_3 and C_4 compose the C_7 . The C_3 has 5 points on C_4 which is therefore of genus 1.

32. Case 6. The C_7 may break up into a C_3 passing through O and a C_4 which have 6 points in common. The genus of C_4 is therefore 0.

Consider a C_4 which is the intersection of $Q \equiv x_1x_2 - x_3x_4 = 0$ and

$$Q' \equiv m_0x_4^2 + (m_1x_1 + m_2x_2 + m_3x_3)x_4 + n_1x_1^2 + n_2x_2^2 + n_3x_3^2 \\ + p_1x_1x_2 + p_2x_1x_3 + p_3x_2x_3 = 0.$$

Eliminate x_4 giving as the projecting cone of C_4 from O ,

$$K_4 \equiv n_3x_3^4 + (p_3x_2 + p_2x_1)x_3^3 + (m_3x_1x_2 + n_1x_1^2 + n_2x_2^2 + p_1x_1x_2)x_3^2 \\ + (m_1x_1^2x_2 + m_2x_1x_2^2)x_3 + m_0x_1^2x_2^2 = 0.$$

Now eliminate x_4 between F_1 and Q and impose the condition that this quintic cone shall contain K_4 . Doing likewise for the other two cubic surfaces of the net we find that the quadrics of the net R have the form

$$H_1 \equiv -m_0x_4^2 + [\alpha_1x_1 - m_2x_2 + (n_2 + c_2 - b_1 - m_3 - p_1)x_3]x_4 + a_1x_1^2 \\ - n_2x_2^2 - n_3x_3^2 + b_1x_1x_2 + (p_3 + d_3 - p_2)x_1x_3 - p_3x_2x_3 = 0,$$

$$H_2 \equiv -m_0x_4^2 + [-m_1x_1 + (\alpha_1 + m_1 - m_2)x_2 \\ + (a_1 + n_1 - d_1 - m_3 - p_1)x_3]x_4 - n_1x_1^2 + c_2x_2^2 - n_3x_3^2 \\ + d_1x_1x_2 - p_2x_1x_3 + d_3x_2x_3 = 0,$$

$$H_3 \equiv (\alpha_1 + m_1 - g_1)x_3x_4 + (p_3 + d_3)x_3^2 + g_1x_1x_2 \\ + (a_1 + n_1)x_1x_3 + (n_2 + c_2)x_2x_3 = 0.$$

33. An interesting special case of (6) occurs when the C_7 breaks up into a C_4 of the second kind and three concurrent lines which are its bisecants.

Let the three concurrent lines be

$$x_1 = 0, \quad x_4 = 0; \quad x_2 = 0, \quad x_4 = 0; \quad x_1 = 0, \quad x_2 = 0.$$

If we impose the condition that these lines lie on every cubic surface of the net, the quadrics have the form

$$H_1 \equiv (\alpha_1x_1 + \alpha_2x_2)x_4 + a_1x_1^2 + b_1x_1x_2 + b_2x_1x_3 = 0,$$

$$H_2 \equiv (\beta_1x_1 + \gamma_3x_2)x_4 + a_1x_1x_2 + b_2x_2x_3 + b_1x_2^2 = 0,$$

$$H_3 \equiv w_0x_4^2 + w_1x_4 + b_2x_3^2 + g_1x_1x_2 + a_1x_1x_3 + b_1x_2x_3 = 0.$$

This completes the cases, given by Montesano, where the C_7 may become composite by breaking up into two pieces, but two other cases are worthy of note. It may happen that the cubic surfaces of the net have a common double point which is not exceptional for the congruence. If the C_7 does not break up and the net has no other peculiarities, this point is quadruple on C_7 whose genus is therefore 1.

Let each cubic surface of the net have a double point at $P \equiv (1, 0, 0, 0)$. The quadrics are then of the form

$$H_1 \equiv u_0 x_4^2 + u_1 x_4 + a_2 x_2^2 + a_3 x_3^2 + b_1 x_1 x_2 + b_2 x_1 x_3 + b_3 x_2 x_3 = 0,$$

$$H_2 \equiv v_0 x_4^2 + (\beta_2 x_2 + \beta_3 x_3) x_4 + c_2 x_2^2 + c_3 x_3^2 + d_3 x_2 x_3 = 0,$$

$$H_3 \equiv w_0 x_4^2 + (\gamma_2 x_2 + \gamma_3 x_3) x_4 + f_2 x_2^2 + g_3 x_2 x_3 = 0.$$

34. Now suppose each quadric of the net R passes simply through O . Then O is a double point on each cubic surface of the net Ξ and a triple point on C_7 , the genus of which is reduced to 3. In this case the conics of Σ can be mapped on a linear congruence as follows:

Establish a $(1, 1)$ correspondence between the planes and the lines of the bundle (O) so that a plane and its related line are in united position. Let $\sum a_i x_i = 0$ be the equation of a plane of the bundle. Let $\sum a'_i x_i = 0$, $\sum a_i x_i = 0$ be the equations of the related line where a'_i is linear in a_i . This relation can be established in ∞^3 ways.

In the plane $\sum a_i x_i = 0$ is a conic whose equation is $\sum a_i H_i = 0$. If the conic passes through O , the line related to the plane cuts the conic apart from O , in only one point G . To get the locus of G , solve the first and third equations for a_i giving $a_i = F_i(x)$, and substitute in the second. Therefore the locus of $F \equiv \sum a_i F_i(x) x_i = 0$, is a monoid of order 4, with vertex at O , and passing simply through C_7 . These monoids form a homoloidal web and hence are suitable for a birational transformation. Two monoids have for variable intersection a conic γ . Therefore a conic of the system would go into the intersection of two planes, that is, would be mapped on a line.

By the linear congruence of conics we have been discussing there is determined on a plane in space an involution reducible to the Geiser type. The congruence had a net of generating surfaces of the third order with a C_7 of genus 5 as directrix.

35. *Other linear congruences.* There are other linear congruences which determine on an arbitrary plane in space an involution reducible to the Geiser type, but every one can be reduced to one or another of the types already discussed. Montesano* gives eleven of these with the transformations by which the reduction can be made.

A linear congruence of conics which determines in a plane an involution reducible to the Bertini type can be obtained by considering a pencil of cubic

* D. Montesano: "Su i varii tipi di congruenze lineari di coniche dello spazio; Note II," *Napoli Rendiconti*, Ser. 3, Vol. 1 (1895).

surfaces having in common 2 double points D, D' . The residual intersection of two surfaces of the pencil is a C_8 of genus 3 with two triple points. The plane which is tangent to a cubic surface of the pencil along DD' contains as residual a line c . Any plane through c cuts the cubic surface containing c in a conic belonging to the congruence. The line c has 2 points on C_8 , hence C_8 has 6 points on each conic. The class of this congruence is 4 and therefore it is of the Bertini type.

By a suitable non-involutorial birational transformation this congruence can be reduced to a congruence of conics of class 0. It follows that the only possible transformations for this case are reducible to the monoidal type.

VI. Criteria for determining the various types of linear congruences of conics in space.

36. In conclusion let us consider some criteria for determining the various types of linear congruences of conics in space. For such a congruence Σ , every surface formed by ∞^1 conics of the system cuts two arbitrary planes α and α' of space (which do not contain conics of the surface and which are not tangent to it) in two curves of the same order and same genus. These curves are respectively invariant in the two involutions which Σ determines on the planes α and α' .

Conversely, consider in an arbitrary plane α , a curve C which is conjugate to itself in an involution j_α which Σ determines on the plane α . The conics of the congruence which cut C are on a surface which cuts another arbitrary plane α' of space in a curve of the same order and genus as C , except for the case when the two planes α and α' are tangent to the surface or contain a conic.

Now it is known that in a plane involution j_α there exists either a pencil of invariant rational curves, or a net of elliptic invariant curves, or only a pencil of elliptic curves. Then the involution is reducible to the Jonquières, Geiser or Bertini type. It is also known that in every case such an invariant curve of the pencil or of the indicated net, corresponds to itself in a rational involution.

Correspondingly, the type is known if a linear congruence of conics has a generating pencil of homoloidal surfaces with rational plane sections; or has a generating net of homoloidal surfaces with elliptic plane sections; or has only a generating pencil of homoloidal surfaces with elliptic plane sections.

37. There are then three different families of linear congruences of conics. For the first kind, if the linear congruence of conics determines on an arbitrary plane in space an involution reducible to the Jonquières type,

it is generated by a pencil of Steiner surfaces or ruled surfaces. For this type the congruence can always be reduced to a bundle of lines, and our transformations to monoidal involutions.

For the congruence of conics of the other two families it is to be noted that a homoloidal surface with elliptic plane sections which contain ∞^1 conics can always be represented on a plane in such a way that the plane sections have for images either a $C_4 : P^2Q^2$ or a $C_3 : P$ having in common the other simple points (in number equal to or less than 5) so that for the congruence in question the order of the generating net or generating pencil formed by the surfaces having elliptic plane sections is equal to or less than 8.

Conversely, given a homoloidal surface S_m ($2 < m < 9$) with elliptic plane sections, it is easy to recognize whether the S_m can be part of a net or of a pencil of surfaces generating a linear congruence of conics, formed by the surfaces having the same order and singularities as S_m .

38. Hence the only types that give rise to a birational involutorial transformation of space that cannot be reduced to the monoidal type are those for which the conics of Σ meet an arbitrary plane in the groups of a Geiser involution. Even then the transformations are non-monoidal only when the conics cut every directrix curve in an even number of points.

CORNELL UNIVERSITY,
June, 1927.

Extensions of Clifford's Chain-Theorem.

By F. MORLEY.

I propose to state more fully what is implied in the final section of my paper "Metric Geometry of the plane n -line," *Transactions of the American Mathematical Society*, Vol. 1 (1900), p. 115. I refer to this as M. G. First, let us recall Clifford's Theorem, *Works*, p. 51.*

In a plane we take lines, say 1, 2, 3, \dots , n . We complete the figure as follows: We mark the intersections 12, \dots . We mark the circles 123, on 12, 23, 31, \dots . There is a point 1234 on the 4 circles 123, 234, 341, 412. There is a circle 12345 on the 5 points 1234, \dots . There is a point 123456 on the 6 circles 12345, \dots . And so on. For an even number of lines the figure ends with a point—the Clifford point; for an odd number with a circle—the Clifford circle.

Regarding the lines as circles on the point ∞ , we have a configuration—the Clifford configuration.

1. *The fundamental curves C^n .* An account of these curves and their osculant theory is given in my paper on Reflexive Geometry,† to which I refer as R. G. I shall state a little differently what is needed here. The aim is to obtain for any given number of lines a curve which plays the part of the circumcircle for three lines.

We take a base circle $|t| = 1$. An equation $x = f(t)$ maps this circle on some curve. The point x is stationary—that is a cusp—of the curve when $dx/dt = 0$.

An especially simple class of curves with $n - 2$ cusps will then be given by

$$(1) \quad dx/dt = \kappa(t - t_1)(t - t_2) \dots (t - t_{n-2}).$$

Denote such a curve by C^n .

Thus when $n = 2$, $dx/dt = \kappa$, $x = \kappa t + x_0$, so that C^2 is a circle. C^3 is a cardioid, and so on.

It is convenient here to mean by C^1 a point, for which $dx/dt = 0$; and by C^0 a line. If $dx/dt = \kappa/t$ and $t = e^{i\theta}$, then $x = x_0 + \kappa\theta$, which denotes a line. If the equation (1) which gives the cusp-parameters is written

$$(1') \quad dx/dt + (n - 1) \{a_1 + (n - 2) a_2 t + \dots + \bar{a}_1 t^{n-2}\} = 0,$$

* For exceptional cases, see W. B. Carver, *American Journal of Mathematics*, Vol. 42 (1920), pp. 137-167.

† *Transactions of the American Mathematical Society*, Vol. 8 (1907), pp. 14-24.

then the lines of the curve C^n are given by

$$(2) \quad x - x_0 + na_1t + \binom{n}{2}a_2t^2 + \cdots + n\bar{a}_1t^{n-1} + (\bar{x} - \bar{x}_0)t^n = 0.$$

We call x_0 the center of the curve. If conversely we write a self-conjugate equation in t , and regard the end-coefficients as conjugate variables, we have a curve C^n with center 0. R. G. § 3.

The fully polarized form of (2) is

$$(3) \quad x - x_0 + a_1s_1 + a_2s_2 + \cdots + (\bar{x} - \bar{x}_0)s_n = 0,$$

where s_1, s_2, \dots, s_n are the product-sums of t_1, t_2, \dots, t_n . This is the osculant line of C^n for the parameters t_i , or points t_i of C^n . If we set $t_1 = t_2 = \cdots = t_m = t$ we have the osculant C^m for the points t_{m+1}, \dots, t_n . Two osculant C^{n-1} 's have a common osculant C^{n-2} and conversely two C^{n-1} 's with a common osculant C^{n-2} are osculants of a C^n . R. G. § 6.

A curve C^n is defined save as to homologies $x = ay + b$ by its $n - 2$ cusp-parameters. Suppose $n = 2m + 1$. Then there are $2m - 1$ cusp-parameters. There is then a canonizant, m parameters apolar (or harmonic) to the cusp-parameters. These m points give an osculant C^{m+1} which is a repeated point. This point is the Clifford point of the C^{2m+1} . The reason for the existence of this point may be stated thus. An osculant C^m is defined by taking $n - m$ points t_i on the given C^n . These have a polar as to the cusp-parameters. This polar gives the cusp-parameters of C^m . Thus when the polar is arbitrary, the cusp-parameters are arbitrary. And this implies that in (1') $a_1 = a_2 = \cdots = 0$. For the curve C^{2m+2} with $2m$ cusps the simplest set apolar to the cusp-parameters is a pencil of sets of $m + 1$ points. Such a set has an osculant C^{m+1} which again is a repeated point. The locus of such points is a circle.* This is the Clifford circle of the C^{2m+2} .

To prove this it will suffice to take one case. Suppose we have a C^4 , with 2 cusps,

$$x - x_0 + 3a_1t + 3\rho t^2 + \bar{a}t^3 = 0.$$

The osculant of t_1, t_2 is the circle

$$x - x_0 + a_1s_1 + \rho s_2 + \bar{a}s_3 = 0.$$

If t_1 and t_2 are apolar or harmonic to

$$a_1 + 2\rho t + \bar{a}t^2 = 0$$

then

$$a_1 + \rho(t_1 + t_2) + \bar{a}t_1t_2 = 0$$

* Strictly, we should speak of the Clifford circle or arc, for the circle may or may not be complete.

and the circle became the point

$$x - x_0 + a_1(t_1 + t_2) + \rho t_1 t_2.$$

If we eliminate $t_1 + t_2$ from these two equations we have

$$\begin{vmatrix} x - x_0 & a_1 \\ a_1 & a\rho \end{vmatrix} = t_1 t_2 \begin{vmatrix} a_1 & \rho \\ \rho & \bar{a} \end{vmatrix}$$

which is a circle (or the arc of a circle). The argument is general—see M. G. § 4, p. 103. We are here restating that § 4 in geometrical language.

Given a C^{2m+2} , each osculant C^{2m+1} has a Clifford point. The locus of these is the Clifford circle. The reason is that if we have $m + 1$ points t_i apolar to $2m$ points τ_i , the polar of t_i has as canonizant the points $t_2 \cdots t_{m+1}$. The algebraic argument is in M. G. § 4.

Given a C^{2m+1} each osculant C^{2m} has a Clifford circle. These are all on the Clifford point. The reason is that if we have m points t_i apolar to $2m - 1$ points, τ_i , the polar of t_1 has an apolar set the m points. The algebraic argument is again in M. G. § 4. The gist is that we regard the Clifford chain as a property of a curve C^n and its osculants. The lines come in by taking $n + 1$ points t_i on C^n . Each n of these points has an osculant line, so that we have a set of $n + 1$ osculant lines. These are any lines and they determine the C^n uniquely (M. G. § 1). This C^n is the fundamental covariant, under the group $x = ay + b$ of homologies, of the $n + 1$ lines. The fundamental covariant C^m of any $m + 1$ of the lines is an osculant of C^n —in fact the osculant for the unused t_i . Thus for two lines the C^1 is the intersection. For three lines the 3 intersections are on the C^2 which is the circumcircle. For four lines, the 4 circumcircles are osculants of the C^3 , and meet at the cusp (the Clifford point). For five lines the 5 C^3 's are osculants of C^4 . The five cusps are on its Clifford circle. And so on.

2. *The incenters of an $(n + 1)$ -line.* A three-line has four inscribed circles. Their centers are the incenters of the three-line. We ask then for the curves C^n which touch $n + 1$ given lines, and more particularly for their centers. These centers we call the *incenters* of the $(n + 1)$ -line.

A line is given most simply by the image in it of the base point 0. Let the images be $x_1, x_2 \cdots x_{n+1}$. We have then from (2) $n + 1$ equations

$$-x_0 + a_1 t_i + \cdots + \bar{a}_1 t_i^{n-1} + (\bar{x}_i - x_0) t_i^n = 0.$$

Eliminating $x_0, a_1 \cdots \bar{a}_1$ we have a determinant

$$\begin{vmatrix} 1 & t_i t_i^2 & \cdots & (\bar{x}_i - \bar{x}_0) t_i^n \end{vmatrix} = 0,$$

and therefore

$$(4) \quad \bar{x}_0 = \sum_{i=1}^{n+1} [\bar{x}_1 t_1^n / (t_1 - t_2) \cdots (t_1 - t_{n+1})].$$

But the clinant of the line given by x_i is $-x_i/\bar{x}_i$ and is from (2) $-t_i^n$. Thus we have

$$(5) \quad x_i = \bar{x}_i t_i^n,$$

so that (4) takes the form, the equation of incenters,

$$(4') \quad \bar{x}_0 = \sum^{n+1} [x_1/(t - t_2) \cdots (t_1 - t_{n+1})].$$

These incenters can be constructed, if we can solve (5) for t_i , that is if we assume that we can divide an angle into n equal parts. This operation is all we need, in addition to Euclid's, in this paper. For any t_i we may substitute ϵt_i where $\epsilon^n = 1$. The set t_i and the set ϵt_i give the same value of x_0 . Hence there are n^n incenters, and n^n inscribed C^n 's.

Each can be named 1, $\epsilon_1, \dots, \epsilon_n$, where $\epsilon_i^n = 1$, and repetitions are allowed.

3. *The axes of an n -line.* The conjugate of (4) is

$$(5) \quad x_0 = \sum [x_1 t_2 \cdots t_{n+1}/(t_2 - t_1) \cdots (t_{n+1} - t_1)].$$

Eliminating x_{n+1} from (4') and (5) we have

$$(6) \quad x_0 \pm \bar{x}_0 t_1 t_2 \cdots t_n = \sum^n [x_1 t_2 \cdots t_n/(t_2 - t_1) \cdots (t_n - t_1)]$$

the sign being $+$ for n odd, $-$ for n even. This is a self-conjugate equation, denoting a line. There are n^{n-1} such lines. We call them the *axes* of the n -lines. They constitute the locus of the centers of inscribed C^n 's. The clinant of an axis is a geometric mean of the clinants of the n lines; in other words if a line makes with a base line an angle θ_i (to the modulus π) an axis makes an angle θ where

$$n\theta = \sum \theta_i.$$

Therefore the n^{n-1} axes fall into n sets of parallel lines, inclined successively at angles π/n . In each set are n^{n-2} parallel lines.

Each axis can be named 1, $\epsilon_1, \dots, \epsilon_n$. If in two axes so named we have the same ϵ in the same place, the axes are parallel.

Two curves C^n touching n lines may or may not have their centers on an axis. Then the common lines of two curves C^n fall into sets. To verify this directly we compare with (2) a second equation

$$(2') \quad x - y_0 + b_1 \tau + \cdots + (\bar{x} - \bar{y}_0) \tau^n = 0$$

and we see at once that if these are to be the same equation we must have $\tau^n = t^n$, $\tau = \epsilon t$. For a selected root of unity ϵ , we have on subtraction an equation of degree n in t giving what we call a *tied* set of n common tangents. This equation is

$$y_0 - x_0 + \cdots + (\bar{y}_0 - \bar{x}_0) t^n$$

so that

$$(y - x_0)/(\bar{y}_0 - \bar{x}_0) = (-)^{n t_1 \cdots t_n}.$$

Accordingly, when the centers of two curves on n lines are on an axis, the n lines are a tied set.

The curves C^n which touch n lines fall then into n^{n-1} discrete systems. The transition from one system to another is when the center falls on two axes. One of the n lines is then a double line of C^n .

It follows that when n is not a prime number the double lines of C^n will fall into sets. For example if $n=4$, the center may be where axes meet at right angles, or where axes meet at $\pi/4$.

If we apply the theory of this section to a triangle abc , we obtain as the locus of centers of inscribed cardioids three sets of three parallel lines, forming equilateral triangles. The vertices of the triangles are the centers of the cardioids which touch a side (say bc) of the given triangle twice. If x_0 be such a center, then the angle x_0bc is a third of the angle abc . For x_0b is an axis of the 3 lines ab and bc twice. Thus if we take the interior trisectors of the angles of a triangle, the points where those adjacent to a side meet form an equilateral triangle.ⁿ

4. *The chain for an incenter.* Taking the equation of an incenter, (4') we interpolate, as in M. G. § 2,

$$(11) \quad \bar{x} = \sum^{n+2} \frac{x_1(t_1 - \tau)}{(t_1 - t_2) \cdots (t_1 - t_{n+2})}$$

this becomes (4') when $\tau = \epsilon_{n+2}$. It is a circle on selected incenters of any $n+1$ lines out of $n+2$.

There are then for $n+2$ lines n^{n+2} circles, each on $n+2$ incenters. For a second interpolation we write

$$(12) \quad \bar{x} = \sum^{n+3} \frac{x_1(t_1 - \tau_1)(t_1 - \tau_2)}{(t_1 - t_2) \cdots (t_1 - t_{n+3})}.$$

This becomes (11) when $\tau_2 = t_{n+3}$. It is an osculant of the curve

$$\bar{x} = \sum \frac{x_1(t_1 - \tau)^2}{(t_1 - t_2) \cdots (t_1 - t_{n+3})}.$$

And the point is that this curve is a C^3 . For differentiating we get as cusp-condition

$$(13) \quad \sum \frac{x_1(t_1 - \tau)}{(t_1 - t_2) \cdots (t_1 - t_{n+3})} = 0,$$

* Morley, *Mathematical Association of Japan for Secondary Mathematics*, Vol. 6, Dec. 1924. This theorem, which I obtained in this way long ago, has excited much interest.

whose conjugate is

$$\sum \frac{x_1(\tau - t)}{(t_2 - t_1) \cdots (t_{n+3} - t_1)} \times t_1 t_2 \cdots t_n = 0.$$

The equation (13) is then self-conjugate.

Thus the circles attached to $n + 2$ out of $n + 3$ lines are osculants of a C^3 , and therefore meet at its cusp. There are then for $n + 3$ lines n^{n+3} cardioids. For $n + 4$ lines, the cardioids are osculants of

$$(14) \quad \bar{x} = \sum \frac{x_1(t - \tau)^3}{(t_1 - t_2) \cdots (t_1 - t_{n+4})}.$$

Writing $d\bar{x}/dt$, and forming its conjugate we see again that (14) has 2 cusps, and is a C^4 . And so on.

We have then the table:

number of lines	1	2	3	4	5	6	...
axes C^0	1	2	3^2	4^3	5^4	6^5	...
incenters C^1		1	2^2	3^3	4^4	5^5	...
circles C^2			1	2^3	3^4	4^5	...
cardioids C^3				1	2^4	3^5	...
C^4					1	2^5	...
C^5						1	...
:							

The table is read diagonally; each C^n is a first osculant of some C^{n+1} in the next column.

The first column says that a line is its own axis.

The second column says that a two-line has two axes and an intersection.

The third column says that a three-line has 3^2 axes; that it has 2^2 incenters, the intersection of the axes of the two-lines contained in it; and that it has one circumcircle, on the intersection of the two-lines.

For the n th column, the axes are new; the incenters arise from the axes of the preceding column, that is the $n(n-1)^{n-2}$ axes of the component $(n-1)$ -lines meet in the $(n-1)^{n-1}$ incenters, there being n axes on a point and $n-1$ points on an axis. The circles arise from the incenters of the preceding column. That is the $n(n-2)^{n-2}$ incenters are on the $(n-2)^{n-1}$ circles, there being n points on each circle, $n-2$ circles on each point. These $n-2$ circles cut at the angle $\pi/(n-2)$. The $(n-3)^{n-1}$ cardioids arise from the $n(n-3)^{n-2}$ circles; the circles are osculants of the cardioids, each C^3 having n osculant C^2 's; and each C^2 osculating $n-3$ C^3 's. And so on.

The leading diagonal indicates Clifford's chain. We notice that the n -line has a unique C^{n-1} .

The second diagonal indicates the chain discussed by F. H. Loud, *Transactions of the American Mathematical Society*.* The ambiguities which enter from (5) are there cleared up by regarding lines as directed. This makes the incenter of a 3-line unique. In general the ambiguities disappear if we recall that two osculants of C^n have a common osculant. Consider the $2^4 C^3$'s for 5 lines. These for 5/6 lines are osculants of C^4 's. The $6 \times 2^4 C^3$'s are osculants of the $2^5 C^4$'s; $6C^3$'s on each C^4 , $2C^4$'s on each C^3 . The C^3 taken from lines 12345 and that taken from 23456 must have a common osculant C^2 (taken from 2345).

The ambiguity is explained in another specific case involving axes by P. S. Wagner in the article following this one.

There is in fact no ambiguity where we name the C^n 's by the roots of unity. The naming is carried on from one column to the next.

This completes the object of this paper, but it is convenient to add a canonical equation of the curve C^n .

4. *Canonical equation of a C^n .* The curve C^n is defined save as to homologies by the $n-2$ cusp-parameters. It is proper to give these intrinsically, so far as possible. When n is odd, say $n=2m+1$, Sylvester pointed out the proper intrinsic or canonical form for an equation, here

$$a_1 + (n-2)a_1t + \dots + \bar{a}_1t^{2m-1} = 0$$

namely the equation

$$(15) \quad \sum^m A_i(t - \tau_i)^{2m-1} = 0.$$

Here the m numbers τ_i give the canonizant, the unique equation of degree m apolar to the given equation. Accordingly the canonical form for a C^{2m+1} is

$$(1/2m)(dx/dt) = \sum^m A_i(t - \tau_i)^{2m-1}$$

or

$$(16) \quad x = \sum^m A_i(t - \tau_i)^{2m}.$$

The equation (15) must be self-conjugate, that is the same as

$$\sum^m \bar{A}_i(t - \tau_i)^{2m-1} / \tau_i^{2m-1} = 0.$$

This is secured by

$$(17) \quad \bar{A}_i = A_i T_i^{2m-1}.$$

The base-point in this canonical form (8) is the Clifford point of any $2m+2$ lines which are an osculant set of the C^{2m+1} . It is in fact the osculant of the canonizant. The simplest case, $m=1$, is the cardioid, with

* Vol. 1 (1900), pp. 323-338.

the canonical equation

$$x = (1 - t)^2.$$

An osculant point is here

$$x = (1 - t_1)(1 - t_2)$$

and 3 such points are on the line

$$x = (1 - t_i)/(1 - t).$$

Four such lines are given by

$$x = \prod^4 [(1 - t_i)/(1 - t)(1 - t')]$$

and are tangents of the parabola

$$x = [\prod^4/(1 - t)^2].$$

And so in general the C^{2m+1} can be immediately connected with

$$x = \sum^m [B_i/(t - \tau_i)^2]$$

which is Clifford's m -fold parabola (*loc. cit.*). The case of C^5 , with 3 cusps, was analysed, with figures, by Father E. C. Phillips (*American Journal*, Vol. 31, 1909). It may be remarked that if the cusps c_i are given there are four curves, for the Clifford point is given by $\sum [1/(x - c_i)^{1/2}] = 0$, which rationalized is a quartic for which $g_2 = 0$.

When n is $2m$, we have for C^{2m} a cusp form of degree $2(m - 1)$. The lowest apolar forms are of degree m , and therefore as in Sylvester's theory we take

$$\frac{1}{2m - 1} \frac{dx}{dt} = \sum^m A_i (t - \tau_i)^{2(m-1)}$$

whence the canonical form is

$$x = \sum^m A_i (t - \tau_i)^{2m-1}.$$

This may be regarded as a first osculant of (8), namely

$$x = \sum^m A_i (t_0 - \tau_i) (t - \tau_i)^{2m-1}.$$

There is no advantage for small values of m . The circle C^2 is naturally best as $x = t$ and the C^4 as $x = 3t - 3\mu t^2 + t^3$.

The canonical form might have been used throughout, but it would not have been possible to use the references.

It is to be remarked that in the form used the coefficients $a_1, a_2 \dots$ of the C^{n-1} of an n -line are invariants of the n -line. They form with the parameters t_i a complete set of rational invariants (under homologies). The parameters are by (3) inversely proportional to the clinants of the lines.

An Extension to Clifford's Chain.*

By PAUL SMITH WAGNER.

1. *Tied Tangents.* Two non-intersecting circles which are exterior to each other have four real tangents which can be separated into two pairs, the one exterior and the other interior. This pairing may also be effected by noticing that the sum of the angles formed by each pair of these tangents with the line of centers is congruent to zero modulo π . We shall define as a set of tangents any pair which fulfils this condition.

Algebraically, if we express these circles as

$$x - 2t + \bar{x}t^2 = 0 \text{ and } x - c - 2\rho\tau + (\bar{x} - \bar{c})\tau^2 = 0,$$

then for a common tangent, we identify the equations so that $\tau^2 = t^2$. We have then $\tau = \pm t$, so that the common tangents fall into two sets. With $\tau = t$, we have

$$c - 2(1 - \rho)t + \bar{c}t^2 = 0,$$

giving one pair of tangents. With $\tau = -t$, we have

$$c + 2(1 - \rho)t + \bar{c}t^2 = 0,$$

giving the other pair.

From these equations it follows that, if the parameters of the tangents in either set are t_1 and t_2 , $t_1 t_2 = c/\bar{c}$, which is the clinant of the line of centers.

Obviously,

$$t_1^2 t_2^2 = (c/\bar{c})^2,$$

which means that if θ_1 , θ_2 and ϕ , respectively, are the angles which the lines t_1 , t_2 and the line of centers make with the axis of reals, $\theta_1 + \theta_2 \equiv 2\phi \pmod{\pi}$. In case we take the axis of reals as the line of centers,

$$\theta_1 + \theta_2 \equiv 0 \pmod{\pi}.$$

Such a set of tangents is a set of *tied* tangents.

This classification leads to an immediate generalization. If we take the two cardioids †

$$(1) \quad x - 3t + 3t^2 - \bar{x}t^3 = 0$$

$$(2) \quad x - c - 3a\tau + 3\bar{a}\tau^2 - (\bar{x} - \bar{c})\tau^3 = 0$$

* W. K. Clifford, *Mathematical Papers*, pp. 51-54.

† F. Morley, *Journal of the Mathematical Society of Japan for Secondary Education*, Vol. 6 (1924).

and identify these equations, we have

$$\tau^3 = t^3.$$

Thus the common lines fall into three sets. If we take $\tau = t$, then

$$c - 3(1-a)t + 3(1-\bar{a})t^2 - \bar{c}t^3 = 0$$

gives the three common lines of one set. For then

$$t_1 t_2 t_3 = c/\bar{c} \text{ and } t_1^3 t_2^3 t_3^3 = (c/\bar{c})^3;$$

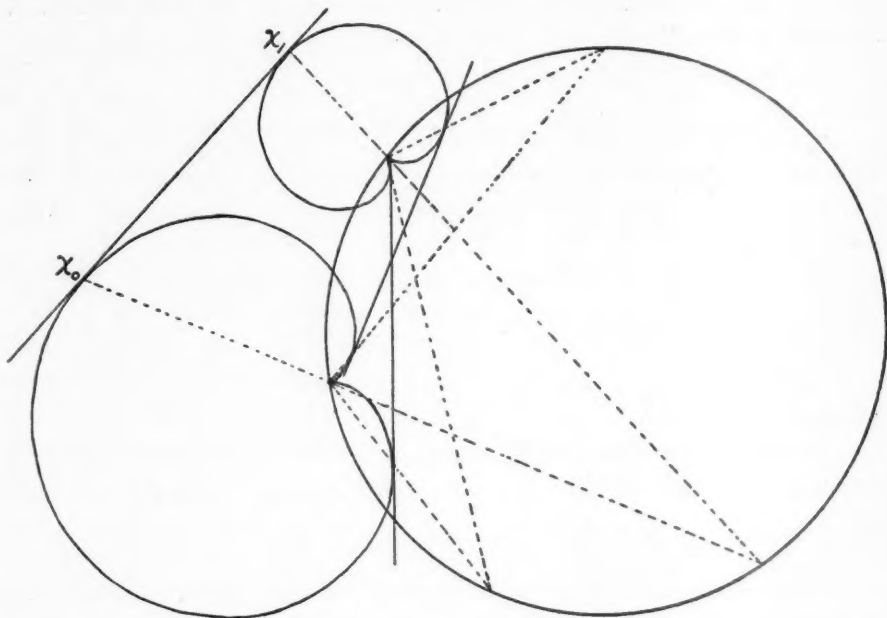


FIG. 1.

so that if θ_i and ϕ , respectively, are the angles which the lines t_i and the line of centers make with the axis of reals,

$$\theta_1 + \theta_2 + \theta_3 \equiv 3\phi \pmod{\pi}.$$

If we take the axis of reals for the line of centers, we have

$$\theta_1 + \theta_2 + \theta_3 \equiv 0 \pmod{\pi}.$$

Here, then, any three tangents with the parameters t_1, t_2, t_3 such that $t_1 t_2 t_3 = c/\bar{c}$, $\omega c/\bar{c}$, or $\omega^2 c/\bar{c}$ are a set of tied tangents.

2. *Method of selecting tied tangents.* The cardioid whose line equation is (1) has $x = 2t - t^2$ for its point equation. At a particular point, this

becomes $x_0 = 2t_1 - t_1^2$ and a line from x_0 through the cusp will have for its equation

$$(3) \quad x = t_1^2 \bar{x} + t_1 + 1.$$

The cardioid (2), which has c for its center, has $x = -\bar{a}\tau^2 + 2a\tau + c$ for its point equation, and, at a particular point, $x_1 = -\bar{a}\tau_1^2 + 2a\tau_1 + c$. The equation of the line from x_1 through the cusp of (2) becomes

$$(4) \quad x = a\tau_1^2 \bar{x} / \bar{a} + f(\tau_1, a, \bar{a}, c, \bar{c}),$$

since the parameter of the cusp is $(a/\bar{a}) + c$ by virtue of the fact that, at the cusp, $dx/d\tau = 0$.

The relative clinant of lines (3) and (4) is $a\tau_1^2/\bar{a}t_1^2$. Accordingly as we let $\tau = t$, ωt or $\omega^2 t$, we have the relative clinant as a/\bar{a} , $\omega a/\bar{a}$ or $\omega^2 a/\bar{a}$, respectively.

If we let $\tau = t$, the line equation of (2) becomes

$$(x - c) - 3at + 3\bar{a}t^2 - (\bar{x} - \bar{c})t^3 = 0$$

and, as shown before, $t_1 t_2 t_3 = c/\bar{c}$ when we consider the common tangents with (1). But when $\tau = t$, x_0 and x_1 will lie on the same tangent line and the cuspidal chords (3) and (4) will have a relative clinant of a/\bar{a} . Hence, this selects three tangents whose cuspidal chords, respectively, form angles such that the relative clinant is a/\bar{a} , so that the three angles are equal, modulo π . Hence, *the three intersections of the cuspidal chords of these three tangents lie on a circle through the cusps.*

But the same is true for $\tau = \omega t$ and $\tau = \omega^2 t$. Therefore, the nine possible tangents of two cardioids are in three sets of three each such that in each set the cuspidal chords intersect on a circle through the cusps. See fig. 1.

Since this puts five points on each circle, the cusps being on each circle, one tangent is sufficient to definitely determine the other two associated with it to complete the set.

The angle of intersection of these circles on the cusps is that of their tangents at the cusps, and hence the angle between a tangent and the join of the cusps of (1) and (2) is, in each case, equal to the angle between the corresponding cuspidal chords. But these angles differ by $2\pi/3$ so that *the three circles cut each other in angles of $2\pi/3$ each.*

3. *Special types of curves.* Two parabolas have at most three common tangents and their corresponding focal chords intersect on their circum-circle. See fig. 2. This can be shown as follows:

$$DAG^* + AGD + GDA = \pi \text{ and } VLY + LYV + YVL = \pi. \text{ But}$$

* Unless otherwise qualified DAG , AGD , etc. stand for the angles DAG , AGD , etc.

$YVL = GDA$ and $LYV = AGD$. Therefore, $DAG = TL\theta = JLI$. Now triangles FCS and JLT are similar for $SFC = TJL$ and $CSF = JTL$. Hence, $FCS = TLJ = \theta LI$. Also, triangles EBM and ILR are similar for $MEB = RIL$ and $BME = IRL$, so that $EBM = RLI = TLJ$. From this it follows that $FCS = EBM$, $DAG + FCS = \pi$ and $DAG + EBM = \pi$. Therefore the points A , B and C are concyclic with θ and θ' .

Expressing the equation of the cardioid in polar form, we have $r = a(1 - \cos \theta)/2$ or $r^{1/2} = a^{1/2} \sin(\theta/2)$. For the parabola we have $r^{-1/2} = a^{-1/2} \sin(\theta/2)$ and for the circle $r = a \sin \theta$. All of these curves come under the general type *

$$(5) \quad r^{p/q} = a^{p/q} \sin(p\theta/q).$$

In the case of the cardioid $p/q = 1/2$, and if we set $p = 1$ and $q = 2$, $p + q = 3$. For the parabola $p/q = -1/2$, so that if $p = -1$, $q = 2$, then $p + q = 1$. In the circle $p/q = 1$ and letting $p = 1$, $q = 1$, we have $p + q = 2$. This seems to indicate that when p/q is in its lowest form, then $|p + q|$ is the number of sets of tied tangents between two curves of the type of (5). It is not meant that only curves of the above type have this tangent relationship.

4. *A theorem on the parabola.* If the parameters of the points 2 and 3 are t_2 and t_3 , respectively, the equation of the tangent 23 is $x + \bar{x}t_2t_3 = t_2 + t_3$ and if τ is the parameter of the focus, the image in 23 is $x + t_2t_3/\tau = t_2 + t_3$. The images of the same focus in the three tangents are on $x + S_3/\tau t = S_1 - t$ which is the equation of the directrix of the parabola, with focus τ , which is tangent to these three lines. This equation can be written more symmetrically as $\tau^2 - x\tau = (\tau - t_1)(\tau - t_2)(\tau - t_3)/(\tau - t)$. The direction of the directrix is $(S_3/\tau)^{1/2}$ and that of any line perpendicular to it is $i(S_3/\tau)^{1/2}$. If we consider the particular perpendicular which passes through the point of contact of t_2t_3 with the parabola, we know that the line t_2t_3 bisects the angle which this makes with the focal chord. That is, if x is the direction of the focal chord, then $[i(S_3/\tau)^{1/2}x]^{1/2} = i(t_2t_3)^{1/2}$ and $x = i(\tau t_2t_3/t_1)^{1/2}$. Now if the parameter of θ [see fig. 2] is τ_1 , then $x = i(\tau_1 t_2t_3/t_1)^{1/2}$, which is the direction of GA . If the parameter of θ' is τ_2 , then $x' = i(\tau_2 t_2t_3/t_1)^{1/2}$ and this is the direction of DA . The difference of direction of GA and DA is $x/x' = (\tau_1/\tau_2)^{1/2}$ which is one-half the central angle τ_1/τ_2 . Therefore, the vertex A is on the circle of which this latter is a central angle. That is, $A, B, C, 1, 2, 3, \theta$ and θ' all lie on the same circle.

* For another curve of this type, see F. Morley, "The Curve of Ambience," *American Journal of Mathematics*, Vol. 46 (1924), pp. 193-200.

Since the direction of θA is $i(\tau_1 t_2 t_3 / t_1)^{1/2}$ and the parameter of θ is τ_1 , the parameter of A must be $t_2 t_3 / t_1$. In like manner, B is $t_1 t_3 / t_2$ and C is $t_1 t_2 / t_3$. These evaluations are on the assumption that the parameters of 1, 2 and 3, respectively, are t_1 , t_2 and t_3 .

It follows that the direction of $1A$ is $i(t_2 t_3)^{1/2}$. But this is the direction of 23. Hence they are parallel. Since a single infinity of parameters satisfy the above condition we have the general theorem: *If a varying parabola has three fixed tangent lines, the focal chord of any one of them passes through a fixed point.*

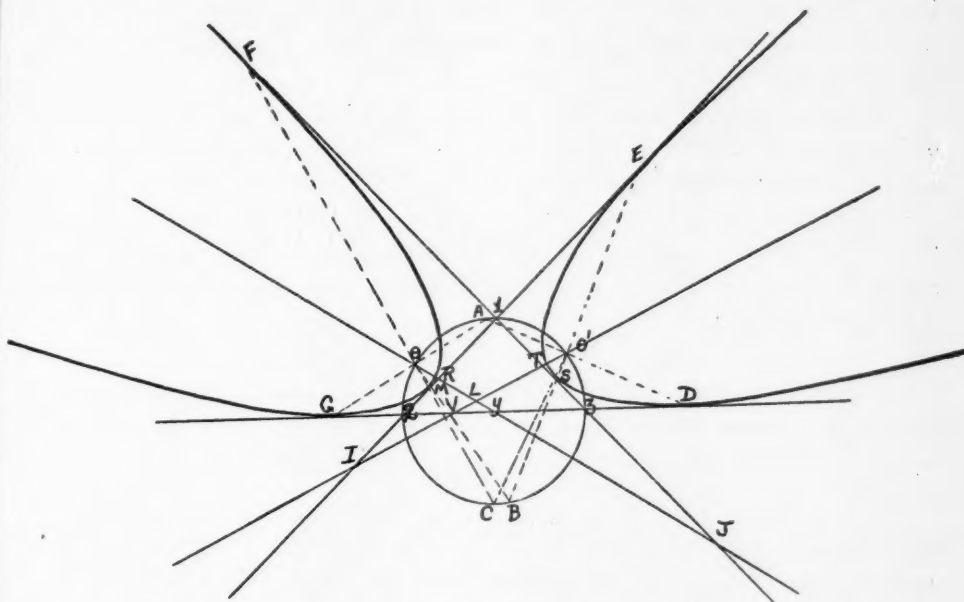


FIG. 2.

5. *Comparison with Professor Loud's work.* In the case of four lines, Professor Loud* showed that the center circles are in two sets, orthogonal systems, of four each obtained from the directed lines as follows:

System I				System II			
+	+	+	+	+	+	+	-
+	+	-	-	+	+	-	+
+	-	-	+	+	-	+	+
-	-	+	+	-	+	+	+

* "Sundry metric theorems concerning N -lines in a plane," *Transactions of the American Mathematical Society*, Vol. 1 (1900), pp. 323-338.

where a change of sign indicates a reversal of direction. Now if we draw our sixteen circles tangent to the four-line, three at a time, and select them

System I								System II				
X	X	X	X	or	X	<	X	<	<	X	X	<
<	<	<	<						X	<	<	X
X	X	X	X						<	X	X	<
<	<	<	<						X	<	<	X

where $<$ denotes an exterior and X an interior pair of tangents, we will obtain the identical grouping of Professor Loud. In each choice there is an even number of each type of tangents—an odd number will not complete the cycle for four points.

In the language of clinants, we can leave the center of any circle in two different ways, namely $t_1 t_2 = c/\bar{c}$ and $t_1 t_2 = -c/\bar{c}$ where the former may be a set of exterior tangents and the latter a set of interior ones. Hence, there are two centers on each circle which gives us a set of eight center circles. That these are in orthogonal pairs is shown by the algebra of Professor Loud's article by replacing t_m by $-t_m$. Hence, Professor Loud's algebra is also the algebra of tied tangents.

6. *Tied tangents applied to cardioids.* There are one hundred and thirty-five cardioids tangent to a five-line, four at a time. Our problem is to effect some systematic classification of these, if possible.

It has already been shown that two cardioids have three sets of three tangents such that if t_1 , t_2 and t_3 are the parameters of any particular set, $t_1 t_2 t_3 = c/\bar{c}$, $\omega c/\bar{c}$ or $\omega^2 c/\bar{c}$. We now select the particular set $t_1 t_2 t_3 = c/\bar{c}$ and proceed by means of this direction to the center of another cardioid of which these three are a set of tied tangents. From this center we proceed by $t_2 t_3 t_4 = c'/\bar{c}'$ to another center, see fig. 3, etc.

We see the $a_1 = t_3 t_4 t_5 / t_2 t_3 t_4 = t_5 / t_2$ and in like manner $a_2 = t_1 t_3$, $a_3 = t_2 / t_4$, $a_4 = t_3 / t_5$ and $a_5 = t_4 / t_1$. Since

$$t_5/t_2 \cdot t_1/t_3 \cdot t_2/t_4 \cdot t_3/t_5 \cdot t_4/t_1 = 1;$$

the polygon is a closed polygon. It is also inscribed in a circle, as we shall show later. We have, then, when we remember that we can leave any center by three ways using $t_1 t_2 t_3 = \omega c/\bar{c}$ or $\omega^2 c/\bar{c}$ instead of c/\bar{c} , this significant theorem:

The centers of the cardioids tangent to a five line, four at a time, arrange themselves so that there are five centers on each circle and three circles on each center. Hence, the centers of the 135 cardioids lie on 81 circles.

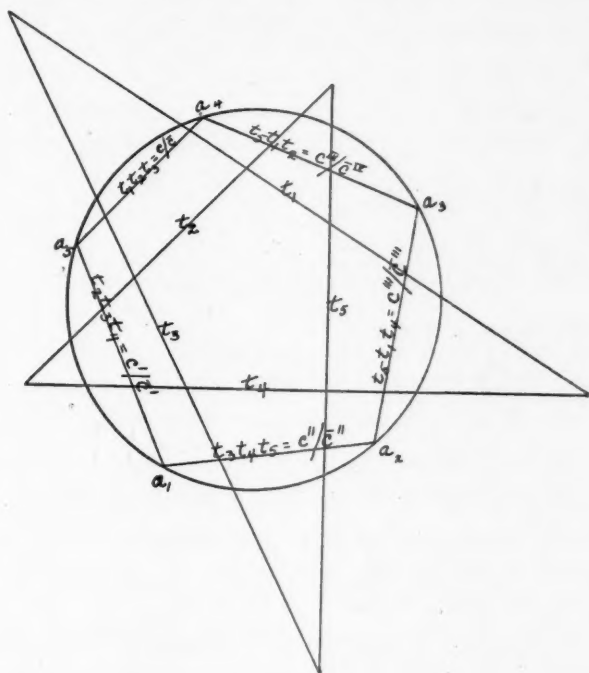


FIG. 3.

7. *The algebraic statement.* In the closing pages of a paper by Professor Morley * the equation of the tangent to a cardioid is given as

$$xt^3 + y = x_0t^3 + y_0 + at^2 + bt$$

which in another form,

$$(y - y_0) - bt - at^2 + (x - x_0)t^3 = 0,$$

is analogous to (2). In fact, if we replace t by $-1/\tau$, we obtain (2). If p is the distance from the origin to this tangent, then

$$p = (Ax + \bar{A}\bar{x} + p)/2(A\bar{A})^{1/2};$$

i. e.,
$$p_1 = [-t_1^3x - y + t_1^3x_0 + y_0 + bt_1 + at_1^2]/2t_1^{3/2}$$

and
$$2p_1t_1^{3/2} = t_1^3x_0 + y_0 + bt_1 + at_1^2.$$

* "Metric Geometry of the Plane N -line," *Transactions of the American Mathematical Society*, Vol. 1 (1900), pp. 97-115.

For four tangents, if x_0 is the center of the cardioid to which these lines are tangent, and if $x_1 = 2p_1 t_1^{3/2}$,

$$x_0 = \sum^4 \frac{x_1}{(t_1 - t_2)(t_1 - t_3)(t_1 - t_4)}$$

If now we put

$$(6) \quad x = \sum \frac{x_1(t_1 - t)}{(t_1 - t_2)(t_1 - t_3)(t_1 - t_4)(t_1 - t_5)}$$

we have a circle whose center is

$$a_1 = \sum^5 \frac{x_1 t_1}{(t_1 - t_2)(t_1 - t_3)(t_1 - t_4)(t_1 - t_5)},$$

and whose radius is $|a_2|$, where

$$a_2 = \sum^5 \frac{x_1}{(t_1 - t_2)(t_1 - t_3)(t_1 - t_4)(t_1 - t_5)},$$

which is the circle on the centers of the five cardioids each one of which (center) is obtained by letting $t = t_1, t_2, t_3, t_4$ and t_5 successively.

The clinant of $a_4 a_5$ is $(a_5 - a_4)/(\bar{a}_5 - \bar{a}_4)$ which we find to be $-1/t_1 t_2 t_3$. But bearing in mind that t_i of this equation is $-1/t_1$ of equation (2) we have $t_1 t_2 t_3$ as clinant of $a_4 a_5$. Hence $t_1^3 t_2^3 t_3^3 = [(a_5 - a_4)/(\bar{a}_5 - \bar{a}_4)]^3$ and $\theta_1 + \theta_2 + \theta_3 \equiv 3\phi \pmod{\pi}$. Hence, the underlying relation of these cardioids whose centers are given by (6) is that they are connected by tied tangents.

8. *The relation of the 3^4 circles.* The method used in establishing this relation is that of Professor Loud. If a_α is the characteristic constant, then

$$a_\alpha = \sum \frac{x_1 t_1^{n-\alpha-3}}{(t_1 - t_2)(t_1 - t_3) \cdots (t_1 - t_n)} \quad (\alpha = 1, 2, \dots, n),$$

and

$$\begin{aligned} \bar{a}_\alpha &= \sum \frac{x_1 t_1^{\alpha+3-n}}{(1/t_1 - 1/t_2) \cdots (1/t_1 - 1/t_n)} \\ &= (-1)^{n-1} \tau \sum \frac{x_1 t_1^{-3} t_1^{n-1} t_1^{\alpha+3-n}}{(t_1 - t_2) \cdots (t_1 - t_n)} \end{aligned}$$

where $\tau = t_1 t_2 t_3 \cdots t_n$. Therefore

$$\begin{aligned} a_\alpha &= (-1)^{n-1} \tau \sum \frac{x_1 t_1^{\alpha-2}}{(t_1 - t_2) \cdots (t_1 - t_n)} \\ &= (-1)^{n-1} \tau a_{n-\alpha-1}. \end{aligned}$$

When we omit the line x_n from the n -line, we have an $(n-1)$ -line whose constants are *

$$a_1 - a_2 t_n, a_2 - a_3 t_n, \dots, a_{n-1} - a_n t_n.$$

Now since $t_1 t_2 t_3 = c/\bar{c}$, or $\omega c/\bar{c}$, or $\omega^2 c/\bar{c}$ then $\omega t_1 t_2 t_3$ and $\omega^2 t_1 t_2 t_3$, on the assumption that $t_1 t_2 t_3 = c/\bar{c}$, take us to the other two centers. That is, this is equivalent to replacing t_n by ωt_n or $\omega^2 t_n$. Three circles derived on these respective assumptions that the n -th line is used or is replaced by ωt_n , or $\omega^2 t_n$, have the equations $x = a_1 - a_2 t$, $x = a_1' - a_2' t$ and $x = a_1'' - a_2'' t$. These three circles coincide at a point which when the n -th line is omitted is called $a_1 - a_2 t_n$, $a_1' - a_2' t_n$ and $a_1'' - a_2'' t_n$. Their angle of intersection is that of the strokes extending from their respective centers. Hence, it is the amplitude of $a_2 t_n/a_2'' t_n$, or a_2/a_2'' , etc.

To obtain this amplitude separate from the ratio of the absolute values, we divide this expression by its conjugate and take the square root of the result. Since the conjugate of a_2 is $(-1)^{n-1} \tau a_{n-3}$ and of a_2'' is $\omega^2 (-1)^{n-1} \tau a_{n-3}'$, we have

$$(a_2/a_2'')^{1/2} (\bar{a}_2/\bar{a}_2')^{1/2} = (a_2/a_2'')^{1/2} (\omega^2 a_{n-3}'/a_{n-3})^{1/2},$$

and in the case of $n = 5$, this becomes ω or $2\pi/3$. Hence, on each of the 135 centers there are three circles cutting each other at an angle of $2\pi/3$.

9. *General conclusions.* As a complete statement, then, we may say that a five line has 135 cardioids, each tangent to a four line, and their centers lie on 3^4 circles which cut each other in threes at these points at an angle of $2\pi/3$ so that there are five centers on each circle and three circles on each center.

* F. Morley, "Metric Geometry of a Plane N -line," *Transactions of the American Mathematical Society*, Vol. 1 (1900), pp. 97-115.

On the Group for a Class of Self-Dual Plane Rational Curves.

BY LUTHER E. WEAR.

1. *Introduction.* We are concerned in this paper with the group of transformations, including both collineations and correlations, under which a curve remains fixed. Self-duality implies that the curve must have the same number of cusps as inflexional tangents, and of double points as it has double lines.

Haskell (*Bulletin of the American Mathematical Society*, January, 1917), discussed the question of the maximum number of cusps of an algebraic plane curve, and enumerated the self-dual curves. The binomial curves,*

$$x_1^n = x_0^{n-r} x_2^r$$

have been studied and shown to be self-dual. The case of the quartic was considered in my dissertation at the Johns Hopkins University,† and the quintic in a later paper.‡

The equations of the Curve. Consider now the case of a rational plane curve of the n th order. Since the class of the curve must equal the order, we have

$$(1) \quad n = n(n-1) - 2d - 3c,$$

where d is the number of double points and c the number of cusps. Also, rational curves being of genus zero must have $(n-1)(n-2)/2$ double points and cusps together, hence,

$$(2) \quad d + c = (n-1)(n-2)/2.$$

Solving equations (1) and (2) for d and c , we find $c = n-2$, $d = (n-2)(n-3)/2$, which are the number of cusps and double points, respectively of the self-dual rational curve.

* Loria, *Spezielle Ebene Kurven*, page 308; Wieleitner, *Algebraische Kurven*, page 136; Snyder, *American Journal of Mathematics*, Vol. 30; Winger, *American Journal of Mathematics*, Vol. 36.

† *American Journal of Mathematics*, April, 1920.

‡ *Bulletin of the American Mathematical Society*, June, 1919.

Similarly for the inflexional tangents,

$$c' = 3n(n-2) - 6[(n-2)(n-3)/2] - 8(n-2),^*$$

which reduces to,

$$c' = n - 2.$$

Also,

$$d' = [n(n-2)(n^2-9)/2] - (2d+3c)(n^2-n-6) \\ + 2d(d-1) + [9c(c-1)/2] + 6dc,$$

which reduces to,

$$d' = (n-2)(n-3)/2$$

giving the number of double lines. Hence the self-dual curve has $n-2$ cusps and also inflexional tangents, while it has $\frac{1}{2}(n-2)(n-3)$ double points with the same number of double lines.

The curve, to be self-dual, must admit a correlation (polarity) or correlations whereby dual singularities are interchanged, and in particular cusps and inflexional tangents are interchanged. The class of curves to be considered here are those which admit of a sufficient number of correlations to send a given cusp into each of the $n-2$ inflexional tangents. The products of these correlations, two at a time, will generate a set of collineations which will permute the cusps. Hence the curve will be self-projective under a cyclic G_{n-2} , or under a larger group containing the G_{n-2} as a subgroup. With the ternary group G_{n-2} , will go a corresponding binary group on the parameter, a g_{n-2} .† The transformations of the cyclic g_{n-2} can be written in the canonical form $t = \epsilon t'$, where $\epsilon^{n-2} = 1$.

The curve then can be written down at once in the following most general way:

$$(3) \quad x_0 = t^n + at^2, \quad x_1 = bt^{n-2} + 1, \quad x_2 = t^{n-1} + t,$$

where with the g_{n-2} on the parameter, there goes a ternary G_{n-2} on the lines, viz.,

$$(4) \quad x_0' = \epsilon^2 x_0, \quad x_1' = x_1, \quad x_2' = \epsilon x_2.$$

It is easily seen that the curve (3) remains unaltered when the transformations $t = \epsilon t'$ and (4) are made. All members of the ternary G_{n-2} can be written,

$$(5) \quad x_0' = \epsilon^{2k} x_0, \quad x_1' = x_1, \quad x_2' = \epsilon^k x_2$$

where $\epsilon^{n-2} = 1$ and $k = 1, 2, \dots, (n-2)$.

* Salmon, *Higher Plane Curves*, page 93.

† See Winger, *American Journal of Mathematics*, January, 1916.

Let us now obtain the line equations of (3) by finding Jacobians two at a time. We have,

$$\begin{aligned}\xi_0 &= -bnt^{2(n-2)} + [(b-1)n^2 + (1-3b)n] t^{n-2} - 1, \\ \xi_1 &= -nt^2\{t^{2(n-2)} - [(a-1)n + 1 - 3a] t^{n-2} + a\}, \\ \xi_2 &= nt\{2bt^{2(n-2)} + [ab(4-n) + n] t^{n-2} + 2a\}.\end{aligned}$$

Since the cusps are to be interchanged cyclically by the g_{n-2} , then they are given by the special set $t^{n-2} - 1$, and the other special set, $t^{n-2} + 1$, will give the inflexional tangents. Since the class must be reduced by the $n-2$ cusps, then $t^{n-2} - 1$ must factor out of ξ_0, ξ_1, ξ_2 . Dividing by this factor and equating the respective remainders to zero we have for $\xi_0, b = n/(n-4)$; for $\xi_1, a = n/(n-4)$; and for $\xi_2, 2(a+b) + 4ab = nab - n$ which is satisfied by $a = b = n/(n-4)$. So the curve may now be written,

$$(3') \quad \begin{aligned}x_0 &= t^n + [n/(n-4)] t^2, & x_1 &= [n/(n-4)] t^{n-2} + 1, \\ x_2 &= t^{n-1} + t, & (n > 4),\end{aligned}$$

and the line equations are,

$$(6) \quad \begin{aligned}\xi_0 &= [n/(n-4)] t^{n-2} - 1, & \xi_1 &= t^n - [n/(n-4)] t^2, \\ \xi_2 &= -[2n/(n-4)] t(t^{n-2} - 1).\end{aligned}$$

2. *The Self-Projective Group.* It is evident now that the curve is also invariant under the transformation $t = 1/t'$ when combined with the collineation,

$$(7) \quad \begin{aligned}x_0' &= x_1, & x_1' &= x_0, & x_2' &= x_2.\end{aligned}$$

If the transformation $t = 1/t'$ be combined with the transformations $t = \epsilon t'$ we obtain those of the form $t = \epsilon/t'$ or in general $t = \epsilon^k/t'$, where $k = 1, 2, \dots, (n-2)$. There is thus generated a dihedral group of order $2(n-2)$ on the parameter t .

If the collineation (7) be combined with (5) we obtain these new collineations:

$$(8) \quad \begin{aligned}x_0' &= \epsilon^{2k} x_1, & x_1' &= x_0, & x_2' &= \epsilon^k x_2.\end{aligned}$$

There will be $n-2$ of these and with the original $n-2$ will constitute a collineation $G_{2(n-2)}$ of which the original cyclic G_{n-2} is a sub-group.

Collineations (4) will leave the reference triangle fixed. Those of the type of (8) are reflexions having as axes the lines,

$$x_0 \pm \epsilon x_1 = 0,$$

and as centers the points $\epsilon : -1 : 0$. The axes of reflexion meet on the point $\xi_2 = 0$ while, dually, the centers lie on the line $x_2 = 0$.

The cusp tangents are given by the determinant,

$$\begin{vmatrix} x_0 & x_1 & x_2 \\ t^n + [n/(n-4)] t^2, & [n/(n-4)] t^{n-2} + 1, & t^{n-1} + t, \\ n(n-1)t^{n-2} + [2n/(n-4)], & [n(n-2)(n-3)/(n-4)] t^{n-4}, & (n-1)(n-2)t^{n-3} \end{vmatrix} = 0,$$

which reduces to,

$$x_0 - \epsilon^2 x_1 = 0.$$

The points of inflexion are $\eta_i^2 : -1 : 0$, where $i = 1, 2, \dots, (n-2)$, and where η_i are the $(n-2)$ -th roots of -1 . Hence the cusp tangents meet at $\xi_2 = 0$, while, dually, the points of inflexion lie on the line $x_2 = 0$. Two cases arise according as n is odd or even: in the first case the flexes are distinct points of intersection of the line $x_2 = 0$ with the curve, since η_i^2 are distinct numbers: in the second case the numbers η_i^2 give only $(n-2)/2$ distinct values and that means points of inflexion will come together in pairs to form bi-flecnodes. Dually, when n is odd the cusp tangents will be distinct, but when n is even they will combine in pairs and we shall have double cusp tangents meeting at $\xi_2 = 0$.

We may summarize the results to this point by saying that in equations (3') we have a curve which is self-projective under a dihedral $G_{2(n-2)}$ of collineations when combined with a dihedral $g_{2(n-2)}$ on the parameter. The elements of the groups may be set out as follows:

$$\begin{array}{l} g_{2(n-2)} \qquad \qquad \qquad G_{2(n-2)} \\ t = \epsilon^k t', \qquad x_0' = \epsilon^{2k} x_0, \quad x_1' = x_1, \quad x_2' = \epsilon^k x_2, \quad \epsilon^{n-2} = 1, \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad k = 1, 2, \dots, (n-1). \\ t = \epsilon^k / t', \qquad x_0' = \epsilon^{2k} x_1, \quad x_1' = x_0, \quad x_2' = \epsilon^k x_2. \end{array}$$

3. *The Self-Dual Group.* Now let us write a transformation which will interchange dual singularities of the curve as, for example, $t\tau = \eta$ where $\eta^{n-2} = -1$. This will interchange the cusp parameter $t = 1$ with the flex parameter $\tau = \eta$ and similarly with the other cusps and flexes. The equation of any line of the curve is,

$$\{-[n/(n-4)] \tau^{n-2} + 1\} x_0 + \{\tau^n - [n/(n-4)] \tau^2\} x_1 + \{[2n/(n-4)] \tau^{n-1} - [2n/(n-4)] \tau\} x_2 = 0,$$

and if we make therein the transformation $t\tau = \eta$ we obtain

$$\{[n/(n-4)] t^{n-2} + 1\} x_0 + \{-\eta^2/t^n - [n/(n-4)] \eta^2/t^2\} x_1 + \{-[2n/(n-4)] \eta/t^{n-1} - [2n\eta/(n-4)] t\} x_2 = 0.$$

Any point of the curve is,

$$\{t^n + [n/(n-4)] t^2\} \xi_0 + \{[n/(n-4)] t^{n-2} + 1\} \xi_1 + [t^{n-1} + t] \xi_2 = 0.$$

If these two polynomials in t are identified there are obtained these equations:

$$(9) \quad \xi_0 = x_0, \quad \xi_1 = -\epsilon^2 \eta^2 x_1, \quad \xi_2 = -[2n/(n-4)] \epsilon \eta x_2.$$

These are the equations of a correlation sending any line of the curve into a point of the curve, and in particular exchanging dual singularities.

If now the correlation (9) is combined with the collineations (5) there arise the $n-2$ correlations,

$$\xi_0 = x_0, \quad \xi_1 = -\epsilon^{2k} \eta^2 x_1, \quad \xi_2 = -[2n/(n-4)] \epsilon^k \eta x_2,$$

where $k = 1, 2, \dots, (n-2)$. Furthermore combining correlation (9) with collineations (8) we obtain a further set of $n-2$ correlations, viz.,

$$\xi_0 = x_1, \quad \xi_1 = -\epsilon^{2k} \eta^2 x_0, \quad \xi_2 = -[2n/(n-4)] \epsilon^k \eta x_2.$$

There have been obtained then the elements of a $G_{4(n-2)}$ of collineations and correlations with a corresponding $g_{4(n-2)}$ on the parameter. It is easily verifiable that the product of any two collineations is a third collineation of the set; that the product of a collineation and a correlation is a correlation of the set; that the product of any two correlations is one of the given collineations; and finally that the inverse of each element is included. We may now tabulate the elements of the group as follows:

Collineations

$$\begin{aligned} t &= \epsilon^k t', & x_0' &= \epsilon^{2k} x_0, & x_1' &= x_1, & x_2' &= \epsilon^k x_2, \\ & & & & & \text{where } \epsilon^{n-2} = 1, & k &= 1, 2, \dots, (n-2), \\ t &= \epsilon^k/t', & x_0' &= \epsilon^{2k} x_1, & x_1' &= x_0, & x_2' &= \epsilon^k x_2. \end{aligned}$$

Correlations

$$\begin{aligned} t\tau &= \epsilon^k \eta, & \xi_0 &= x_0, & \xi_1 &= -\epsilon^{2k} \eta^2 x_1, & \xi_2 &= -[2n/(n-4)] \epsilon^k \eta x_2, \\ & & & & & \text{where } \eta^{n-2} = -1, \\ t &= (\epsilon^k/\eta)\tau, & \xi_0 &= x_1, & \xi_1 &= -\epsilon^{2k} \eta^2 x_0, & \xi_2 &= -[2n/(n-4)] \epsilon^k \eta x_2. \end{aligned}$$

Among the correlations only the first set written will be polarities.

4. *Special Case of the Sextic.* The foregoing general theory may be illustrated by the ρ^6 . The equations of the curve are

$$x_0 = t^6 + 3t^2, \quad x_1 = 3t^4 + 1, \quad x_2 = t^5 + t.$$

The cusps are given by $t^4 - 1$ and the points of inflexion by $t^4 + 1$. On $\xi_2 = 0$ we have double cusp tangents and on $x_2 = 0$ bi-flecnodes. Using the general results as given above we may write the self-dual group of the ρ^6 as follows:

Collineations.

$t = t'$	$x_0' = x_0,$	$x_1' = x_1,$	$x_2' = x_2$	(1)
$t = it'$	$x_0' = -x_0,$	$x_1' = x_1,$	$x_2' = ix_2$	(S)
$t = -t'$	$x_0' = x_0,$	$x_1' = x_1,$	$x_2' = -x_2$	(S ²)
$t = -it'$	$x_0' = -x_0,$	$x_1' = x_1,$	$x_2' = -ix_2$	(S ³)
$t = 1/t'$	$x_0' = x_1,$	$x_1' = x_0,$	$x_2' = x_2$	(T)
$t = i/t'$	$x_0' = -x_1,$	$x_1' = x_0,$	$x_2' = ix_2$	(ST)
$t = -1/t'$	$x_0' = x_1,$	$x_1' = x_0,$	$x_2' = -x_2$	(S ² T)
$t = -i/t'$	$x_0' = -x_1,$	$x_1' = x_0,$	$x_2' = -ix_2$	(S ³ T)

Correlations

$t = i^{1/2}/\tau,$	$\xi_0 = x_0,$	$\xi_1 = ix_1,$	$\xi_2 = -6(i)^{1/2}x_2,$	(π)
$t = -(i)^{1/2}/\tau,$	$\xi_0 = -x_0,$	$\xi_1 = ix_1,$	$\xi_2 = 6(i)^{1/2}x_2,$	(πS)
$t = (i)^{1/2}/\tau,$	$\xi_0 = x_0,$	$\xi_1 = ix_1,$	$\xi_2 = 6(i)^{1/2}x_2,$	(πS^2)
$t = -(i)^{1/2}/\tau,$	$\xi_0 = -x_0,$	$\xi_1 = ix_1,$	$\xi_2 = 6(-1)^{1/2}x_2,$	(πS^3)
$t = i^{1/2}\tau,$	$\xi_0 = x_1,$	$\xi_1 = ix_0,$	$\xi_2 = -6(i)^{1/2}x_2,$	($\pi \cdot T$)
$t = -(i)^{1/2}\tau,$	$\xi_0 = -x_1,$	$\xi_1 = ix_0,$	$\xi_2 = -6(-i)^{1/2}x_2,$	($\pi \cdot ST$)
$t = (i)^{1/2}\tau,$	$\xi_0 = x_1,$	$\xi_1 = ix_0,$	$\xi_2 = 6(i)^{1/2}x_2,$	($\pi \cdot S^2T$)
$t = -(i)^{1/2}\tau,$	$\xi_0 = -x_1,$	$\xi_1 = ix_0,$	$\xi_2 = 6(-i)^{1/2}x_2,$	($\pi \cdot S^3T$)

5. *Perspective* ρ^{n-1} . We proceed now to find the equations of a curve of the $(n-1)$ -th order which shall be perspective to the ρ^n i. e. a curve such that a line t of the $(n-1)$ -ic shall cut out the point t on the ρ^n . A line joining the points t_1 and t_2 of the ρ^n can be written

$$\begin{vmatrix} x_0 & x_1 & x_2 \\ t_1^n + [n/(n-4)] t_1^2, & [n/(n-4)] t_1^{n-2} + 1, & t_1^{n-1} + t_1 \\ t_2^n + [n/(n-4)] t_2^2, & [n/(n-4)] t_2^{n-2} + 1, & t_2^{n-1} + t_2 \end{vmatrix} = 0,$$

which becomes, after amplifying and factoring out $t_1 - t_2$,

$$\begin{aligned} & \{ [n/(n-4)] t_1^{n-2} t_2^{n-2} - [n/(n-4)] t_1 t_2 (t_1^{n-4} + t_1^{n-5} t_2 + \dots + t_2^{n-4}) \\ & + (t_1^{n-2} + t_1^{n-3} t_2 + \dots + t_2^{n-2}) + 1 \} x_0 \\ & + \{ t_1^{n-1} t_2^{n-1} + t_1 t_2 (t_1^{n-2} + t_1^{n-3} t_2 + \dots + t_2^{n-2}) \\ & - [n/(n-4)] t_1^2 t_2^2 (t_1^{n-4} + t_1^{n-5} t_2 + \dots + t_2^{n-4}) + [(n/(n-4))] t_1 t_2 \} x_1 \\ & + \{ -[n/(n-4)] t_1^{n-2} t_2^{n-2} (t_1 + t_2) + [n/(n-4)] 2 t_1^2 t_2^2 (t_1^{n-5} \\ & + t_1^{n-6} t_2 + \dots + t_2^{n-5}) - (t_1^{n-1} + t_1^{n-2} t_2 + \dots + t_2^{n-1}) \\ & - [n/(n-4)] (t_1 + t_2) \} x_2 = 0. \end{aligned}$$

This may be arranged as a polynomial in t_1 as follows:

$$\begin{aligned}
 & \{ (t_2^{n-1} + t_2) x_1 - [n/(n-4)] t_2^{n-2} + 1 \} x_2 \} t_1^{n-1} + \{ [n/(n-4)] t_2^{n-2} x_0 + x_0 \\
 & - [n/(n-4)] t_2^2 x_1 + t_2^2 x_1 - [n/(n-4)] t_2^{n-1} x_2 - t_2 x_2 \} t_1^{n-2} \\
 & + \{ - [n/(n-4)] t_2 x_0 + t_2 x_0 - [n/(n-4)] t_2^3 x_1 + t_2^3 x_1 \\
 & + [n/(n-4)] t_2^2 x_2 - t_2^2 x_2 \} t_1^{n-3} + \{ - [n/(n-4)] t_2^2 x_0 + t_2^2 x_0 \\
 & - [n/(n-4)] t_2^4 x_1 + t_2^4 x_1 + [n/(n-4)] t_2^2 x_2 - t_2^3 x_2 \} t_1^{n-4} \\
 (10) \quad & + \dots \\
 & + \{ - [n/(n-4)] t_2^{n-4} x_0 + t_2^{n-4} x_0 - [n/(n-4)] t_2^{n-2} x_1 + t_2^{n-2} x_1 \\
 & + [n/(n-4)] t_2^{n-3} x_2 - t_2^{n-3} x_2 \} t_1^2 + \{ - [n/(n-4)] t_2^{n-3} x_0 \\
 & + t_2^{n-3} x_0 + [n/(n-4)] t_2 x_1 + t_2^{n-1} x_1 - [n/(n-4)] x_2 - t_2^{n-2} x_2 \} t_1 \\
 & + \{ x_0 (t_2^{n-2} + 1) - [n/(n-4)] t_2 x_2 - t_2^{n-1} x_2 \} = 0.
 \end{aligned}$$

Given a point t_2 equation (10) gives the $n-1$ points t_1 in which the line through t_2 cuts the curve. Thus an involution, $I_{n-1,1}$ is set up on the curve, the fixed points of which may be written,

$$(11) \quad a_0 t^{n-1} + a_1 t^{n-2} + \dots + a_{n-2} t + a_{n-1} = 0.$$

The fixed points are apolar to every set; hence writing the condition that (11) be apolar to (10) we have,

$$\begin{aligned}
 & a_0 \{ x_0 (t^{n-2} + 1) - t (t^{n-2} + [n/(n-4)]) x_2 \} \\
 & - a_1 \{ - [n/(n-4)] t^{n-3} x_0 + t^{n-3} x_0 + [n/(n-4)] t x_1 + t^{n-1} x_1 \\
 & - [n/(n-4)] x_2 - t^{n-2} x_2 \} + a_2 \{ - [n/(n-4)] t^{n-4} x_0 + t^{n-4} x_0 \\
 & - [n/(n-4)] t^{n-2} x_1 + t^{n-2} x_1 + [n/(n-4)] t^2 t^{n-3} x_2 - t^{n-3} x_2 \} \\
 & + a_3 \{ - [n/(n-4)] t^{n-5} x_0 + t^{n-5} x_0 - [n/(n-4)] t^{n-3} x_1 \\
 & + t^{n-3} x_1 + [n/(n-4)] t^2 t^{n-4} x_2 - t^{n-4} x_2 \} \\
 & + \dots \\
 & + (-1)^{n-1} a_{n-3} \{ - [n/(n-4)] t x_0 + t x_0 - [n/(n-4)] t^3 x_1 \\
 & + t^3 x_1 + [n/(n-4)] t^2 t^2 x_2 - t^2 x_2 \} - (-1)^{n-1} a_{n-2} \{ [n/(n-4)] t^{n-2} x_0 \\
 & + x_0 - [n/(n-4)] t^2 x_1 + t^2 x_1 - [n/(n-4)] t^{n-1} x_2 - t x_2 \} \\
 & + (-1)^{n-1} a_{n-1} \{ (t^{n-1} + t) x_1 - [n/(n-4)] t^{n-2} + 1 \} x_2 \} = 0.
 \end{aligned}$$

Since the parameter t runs over the curve, it is seen that the last term consists of the factors a_{n-1} and $x_2 x_1 - x_1 x_2$, the latter of which vanishes identically. Collecting coefficients of x_0 , x_1 , x_2 , and equating to ξ_0 , ξ_1 , ξ_2 , respectively, we have,

$$\begin{aligned}
 (12) \quad \xi_0 &= \{ a_0 - (-1)^{n-1} [n a_{n-2}/(n-4)] \} t^{n-2} + [4 a_1/(n-4)] t^{n-3} \\
 & - [4 a_2/(n-4)] t^{n-4} + \dots + (-1)^n [4 a_{n-3}/(n-4)] t + [a_0 - (-1)^{n-1} a_{n-2}],
 \end{aligned}$$

$$\begin{aligned}\xi_0 &= -a_1 t^{n-1} - [4a_2/(n-4)] t^{n-2} + [4a_3/(n-4)] t^{n-3} \\ &\quad + \cdots + (-1)^{n-1} [4a_{n-2}/(n-4)] t^2 - [na_1/(n-4)] t, \\ \xi_2 &= \{-a_0 + (-1)^{n-1} [na_{n-2}/(n-4)]\} t^{n-1} + a_1 t^{n-2} \\ &\quad + [8(n-2)/(n-4)^2] a_2 t^{n-3} - [8(n-2)/(n-4)^2] a_3 t^{n-4} \\ &\quad + \cdots + (-1)^{n-1} [8(n-2)/(n-4)^2] a_{n-3} t^2 \\ &\quad + \{-[na_0/(n-4)] + (-1)^{n-1} a_{n-2}\} t + [na_1/(n-4)].\end{aligned}$$

Or the equations may be written more briefly as follows:

$$\begin{aligned}\xi_0 &= \{a_0 - (-1)^{n-1} [na_{n-2}/(n-4)]\} t^{n-2} \\ &\quad + [4/(n-4)] s \sum_3^{n-1} (-1)^{s-1} a_{s-2} t^{n-s} + [a_0 - (-1)^{n-1} a_{n-2}], \\ (12) \quad \xi_1 &= -a_1 t^{n-1} - [4/(n-4)] s \sum_3^{n-1} (-1)^{s-1} a_{s-1} t^{n-s+1} - [na_1/(n-4)] t, \\ \xi_2 &= \{-a_0 + (-1)^{n-1} [na_{n-2}/(n-4)]\} t^{n-1} + a_1 t^{n-2} \\ &\quad + [8(n-2)/(n-4)^2] s \sum_3^{n-2} (-1)^{s-1} a_{s-1} t^{n-s} \\ &\quad + \{-[na_0/(n-4)] + (-1)^{n-1} a_{n-2}\} t + [na_1/(n-4)].\end{aligned}$$

It is easily to be verified that, if we take the x 's from the ρ^n and the ξ 's as written for r^{n-1} , the row product $(x\xi)$ vanishes identically, which means incidence of a point of ρ^n with a line of r^{n-1} .

Among the indefinite number of curves represented by equations (12) there will be a unique curve invariant under the collineation group, i. e. self-projective. The ρ^n as given in lines by equations (6), is invariant under the transformations $t = \epsilon t'$ when combined with,

$$\xi_0' = \xi_0, \quad \xi_1' = \epsilon^2 \xi_1, \quad \xi_2' = \epsilon \xi_2.$$

We require that the perspective curves (12) be invariant under the same transformations. Since the equation for ξ_0 is to be unchanged then ξ_0 can contain only a constant term and t^{n-2} since $\epsilon^{n-2} = 1$; also the equation for ξ_1 can contain only t^2 since ϵ^2 must factor out; and finally, ξ_2 must contain only t^{n-1} and t since ϵ factors out. Hence the curve may now be written,

$$\begin{aligned}(13) \quad \xi_0 &= \{a_0 - (-1)^{n-1} [na_{n-2}/(n-4)]\} t^{n-2} + [a_0 - (-1)^{n-1} a_{n-2}], \\ \xi_1 &= [4/(n-4)] (-1)^{n-1} a_{n-2} t^2, \\ \xi_2 &= \{-a_0 + (-1)^{n-1} [na_{n-2}/(n-4)]\} t^{n-1} \\ &\quad + \{-[na_0/(n-4)] + (-1)^{n-1} a_{n-2}\} t.\end{aligned}$$

Furthermore the curve is to be invariant under $t = 1/t'$, with which go the transformations,

$$(14) \quad \xi_0' = \xi_1, \quad \xi_1' = \xi_0, \quad \xi_2' = \xi_2.$$

Making the substitution $t = 1/t'$ in (13) and clearing of fractions we have,

$$\begin{aligned}\xi_0' &= \{a_0 - (-1)^{n-1} [na_{n-2}/(n-4)]\}t + [a_0 - (-1)^{n-1}a_{n-2}]t^{n-1}, \\ \xi_1' &= [4/n-4](-1)^{n-1}a_{n-2}t^{n-3}, \\ \xi_2' &= \{-a_0 + (-1)^{n-1} [na_{n-2}/(n-4)]\} \\ &\quad + \{-[na_0/(n-4)] + (-1)^{n-2}a_{n-2}\}t^{n-2}.\end{aligned}$$

It may be seen that conditions (14) are satisfied when the coefficient of t^{n-1} is made to vanish, i. e. $a_0 = (-1)^{n-1}a_{n-2}$. Hence the equations may be written,

$$\xi_0 = t^{n-3}, \quad \xi_1 = -t, \quad \xi_2 = -t^{n-2} + 1.$$

The curve has now become one of order $n-2$ instead of $n-1$.

To illustrate the matter of perspective curves consider first the ρ^5 . The general set of perspective $(n-1)$ -ics become when n equals 5,

$$\begin{aligned}\xi_0 &= (a_0 - 5a_3)t^3 + 4a_1t^2 - 4a_2t + (a_0 - a_3), \\ \xi_1 &= -a_1t^4 - 4a_2t^3 + 4a_3t^2 - 5a_1t, \\ \xi_2 &= (-a_0 + 5a_3)t^4 + a_1t^3 + 24a_2t^2 + (-5a_0 + a_3)t + 5a_1.\end{aligned}$$

The equations of the self-dual ρ^5 are,

$$x_0 = t^5 + 5t^2, \quad x_1 = 5t^3 + 1, \quad x_2 = t^4 + t.$$

It is easily verified that $(x\xi)$ vanishes. The unique perspective curve invariant under the group is,

$$\xi_0 = t^2, \quad \xi_1 = -t, \quad \xi_2 = -t^3 + 1.$$

In the case of the ρ^6 we have for the general set,

$$\begin{aligned}\xi_0 &= (a_0 + 3a_4)t^4 + 2a_1t^3 - 2a_2t^2 + 2a_3t + (a_0 + a_4), \\ \xi_1 &= -a_1t^5 - 2a_2t^4 + 2a_3t^3 - 2a_4t^2 - 3a_1t, \\ \xi_2 &= (-a_0 - 3a_4)t^5 + a_1t^4 + 8a_2t^3 - 8a_3t^2 + (-3a_0 - a_4)t + 3a_1.\end{aligned}$$

and the unique perspective curve invariant under the group is,

$$\xi_0 = t^3, \quad \xi_1 = -t, \quad \xi_2 = -t^4 + 1.$$

Determination of All the Abstract Groups of Order 72.

By G. A. MILLER.

A group G of order 72 may contain either one or four subgroups of order 9, and the number of its subgroups of order 8 must be one of the following three numbers: 1, 3, 9. If it contains only one subgroup of each of the two orders 9 and 8, it must be a direct product of these two subgroups. Since there are two groups of order 9 and five groups of order 8, there are ten distinct such direct products. These ten groups include the six possible abelian groups of order 72. The four non-abelian groups included in this category are the direct products of the two possible groups of order 9 and either the octic group or the quaternion group.

I. *One subgroup of order 9 and three subgroups of order 8.* If G contains three subgroups of order 8, it must transform them according to the symmetric group of degree 3 since the Sylow subgroups of every group must be transformed according to a non-regular substitution group whose order is divisible by the prime number which divides the order of these Sylow subgroups. Hence such a G contains an invariant subgroup of index 2 corresponding to the subgroup of index 2 in the symmetric group of degree 3. As this invariant subgroup contains only one subgroup of each of the orders 9 and 4, it is abelian. Its Sylow subgroup of order 9 must be non-cyclic since it must admit an automorphism in which just three of the operators of the subgroup of order 9 correspond to themselves and one of the other operators of this subgroup is transformed into its inverse. Hence such a group is the direct product of a group of order 3 and a group of order 24 which involves three subgroups of order 8 and only one subgroup of order 3. As there are seven such groups of order 24, there are exactly seven groups of order 72 which contain only one subgroup of order 9 and three subgroups of order 8.

II. *One subgroup of order 9 and nine subgroups of order 8.* The nine subgroups of order 8 are transformed under G according to a transitive group of degree 9 whose order is 18, 36, or 72. If this order is 18, G contains an abelian invariant subgroup of order 36, and it may be constructed by ad-

joining to this subgroup an operator whose order is a power of 2 and which transforms every operator of the Sylow subgroup of order 9 into its inverse. There are eight such transformations, and two groups correspond to each of six of these while only one group corresponds to each of the other two. Hence there are 14 such distinct groups of order 72.

When the nine subgroups of order 8 are transformed under G according to a group of order 36, the Sylow subgroup of order 9 contained in G must be transformed according to a subgroup of order 4 in its group of isomorphisms, and hence this Sylow subgroup must be non-cyclic. The group of isomorphisms of this Sylow subgroup contains a cyclic and a non-cyclic set of conjugate subgroups of order 4 and each of these subgroups involves the operator of order 2 which transforms every operator into its inverse. Hence G involves either the generalized dihedral or the generalized dicyclic subgroup of order 36. The group of isomorphisms of each of these subgroups is known to be the holomorph of the abelian group of order 18 contained therein* and hence it contains 18 operators which transform the operators of this abelian subgroup in the same way.

As this abelian subgroup is transformed according to a group of order 2 under the group of inner isomorphisms of the said groups of order 36, and as the 18 isomorphisms of order 4 which transform this abelian subgroup in the same manner appear in two sets of conjugates, while the six isomorphisms of order 2 which transform this abelian group in the same way and are not inner isomorphisms also appear in two sets of conjugates under the group of isomorphisms of the given groups of order 36, we have to consider only three automorphisms of each of these two groups. In the case of the generalized dihedral group of order 36, the automorphism of order 2 under which six non-invariant operators of order 2 correspond to themselves evidently gives rise to two groups in which the subgroups of order 8 are abelian and of type $(1, 1, 1)$ and $(2, 1)$ respectively. The automorphism of order 2 under which no non-invariant operator of order 2 corresponds to itself gives rise to only one group, while the automorphism of order 4 gives rise to an additional group of order 72. Hence there are four such groups in this case. As there are also four such groups which involve the generalized dicyclic group of order 36, there are eight groups of order 72 which satisfy the conditions that each of them contains only one subgroup of order 9 and nine subgroups of order 8 and transforms these nine subgroups according to a group of order 36.

* Miller, Blichfeldt, Dickson, *Finite Groups* (1916), pp. 169 and 170.

When the nine subgroups of order 8 are transformed according to a group of order 72, the groups must be simply isomorphous with a transitive group of degree 9 and of order 72. It is well known that there are three such groups and that they are abstractly distinct.* Hence the total number of groups of order 72 which separately involve only one subgroup of order 9 and nine subgroups of order 8 is 25. It will be found that this is exactly half the number of the distinct groups of order 72.

III. *Four subgroups of order 9.* The four subgroups of order 9 must be transformed under G either according to the alternating or according to the symmetric group of degree 4. In the former case, each subgroup of order 8 contained in G must transform the four subgroups of order 9 according to the non-cyclic regular group of order 4, while in the latter case, these four subgroups are transformed under a subgroup of order 8 according to the octic group. In each case, these four subgroups have a cross-cut of order 3 which is invariant under G . When these subgroups are transformed under G according to the alternating group of degree 4, then G involves an invariant subgroup of order 24 which contains only one subgroup of order 3. If this subgroup contains also only one subgroup of order 8, the latter subgroup must admit an automorphism of order 3 and hence it is either the quaternion group or the abelian group of type $(1, 1, 1)$. There are four such groups of order 72. Two of them involve operators of order 9 while the other two do not have this property.

If the given subgroup of order 24 contains three subgroups of order 8, it must involve as an invariant subgroup the symmetric group of order 6. As this is a complete group, the subgroup of order 24 must be the direct product of this group of order 6 and the non-cyclic group of order 4. Hence G must involve a subgroup of order 12 composed of all its operators which are commutative with every operator of the said invariant symmetric group of order 6, and it must therefore be the direct product of the tetrahedral group and this symmetric group. It remains therefore to determine the groups of order 72 which involve four subgroups of order 9, and transform these subgroups according to the symmetric group of degree 4. If such a group contains no operator of order 9, it must involve two operators of order 3 whose product is of order 2, and hence it must contain as an invariant subgroup the direct product of the tetrahedral group and a group of order 3. To this direct product we may add two operators of order 2 which transform the tetrahedral group according to an outer isomorphism.

* F. N. Cole, *Bulletin of the American Mathematical Society*, Vol. 2 (1892), p. 250.

One of these operators is commutative with the invariant operator of order 3 in the given direct product while the other transforms this operator into its inverse. Hence there are two distinct groups of order 72 which involve this direct product and transform its four subgroups of order 9 according to the symmetric group of order 24.

When the four subgroups of order 9 contained in G are cyclic and are transformed under G according to the symmetric group of degree 4, the subgroup of order 36 which corresponds to the alternating group of degree 4 involves two operators s_1, s_2 of order 9 which satisfy the following two conditions:

$$s_1^3 = s_2^{-3}, \quad (s_1 s_2)^2 = 1.$$

These equations represent a generalization of those which define the tetrahedral group. It is easy to prove that when s_1 and s_2 are two non-commutative operators which satisfy these two equations then 1, $s_1 s_2, s_2 s_1, s_1^{-1} s_2 s_1^2$ is the non-cyclic group of order 4, and this is the commutator subgroup of the group generated by s_1, s_2 . This subgroup of order 4 and s_1^3 generate an invariant abelian subgroup of index 3 under the group generated by s_1, s_2 . Moreover, the commutator quotient group is always cyclic and these generators may be so chosen that the order of this invariant subgroup is an arbitrary multiple of 4. In the present case this order is 12 and the order of all the additional operators in the group generated by s_1, s_2 is 9. In general, there is one and only one such group whose order is an arbitrary multiple of 12. Since an operator of order 2 which is adjoined to the given group of order 36 to obtain the group of order 72 transforms two of the four cyclic subgroups of order 9 into themselves and is not commutative with their generators, it must transform their operators into their inverses, and hence the automorphism of the given subgroup of order 36 is completely determined. There are therefore eight groups of order 72 which contain separately four subgroups of order 9 and the total number of groups of order 72 is 50. Each of these groups is obviously solvable. They may be classified as follows:

Number of subgroups of order 8....	1	3	9	1	3
Number of subgroups of order 9....	1	1	1	4	4
Number of groups of order 72.....	10	7	25	4	4

3
to
ve
ng
:
re
lie
4
wo
:
:
a-
a-
is
of
te
py
se
is
ne
9.
ry
en
ne
ve
es,
ly
in
of
be